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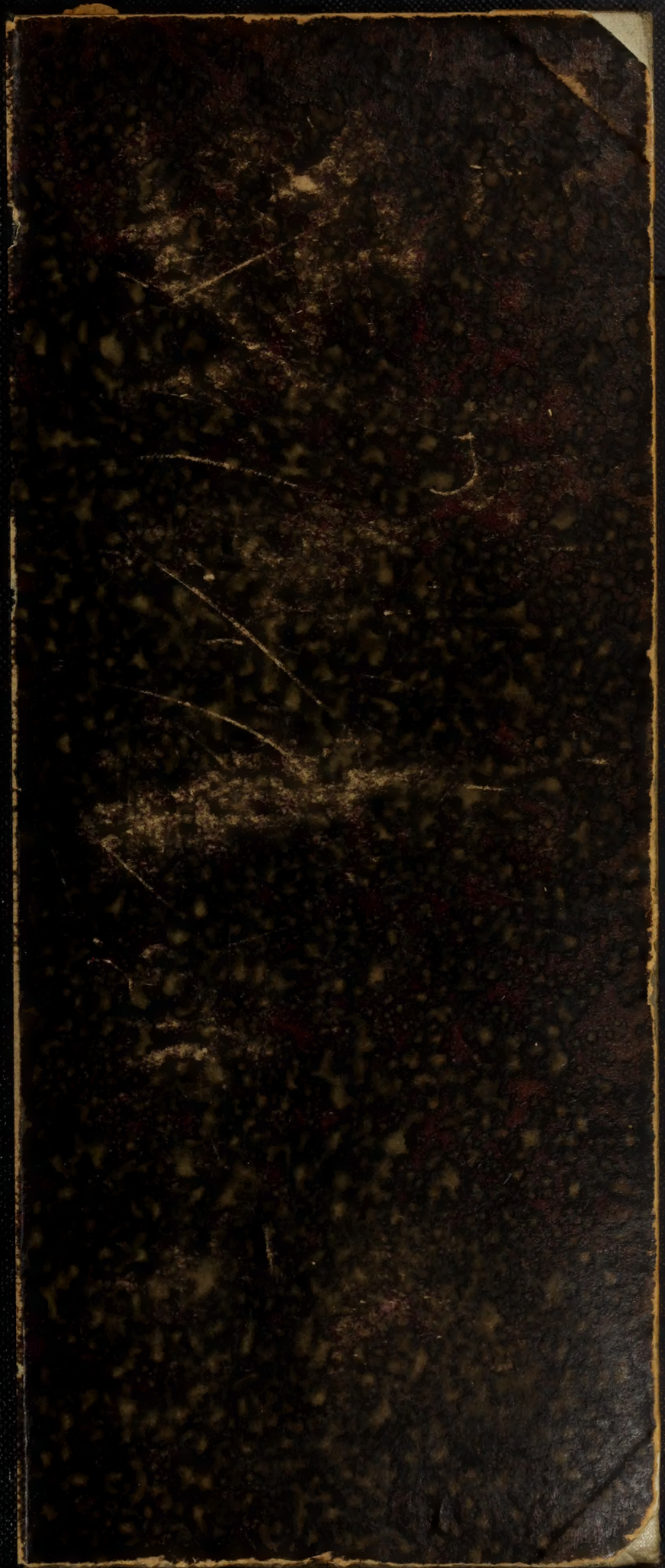

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GENERAL PROPERTIES OF ASYMPTOTIC EXPANSIONS

BY

WALTER JOHN RISLEY, B.S., (MICH.) '00

THESIS FOR THE DEGREE MASTER OF ARTS
IN THE GRADUATE SCHOOL
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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mr. Walter John Risley

ENTITLED General Properties of Asymptotic Expansions

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Master of Arts

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Introduction.

1. We generally agree to date the beginning of modern analysis from the time of Abel (1807-1829) and Cauchy (1789-1857). The work of these men is characterized throughout by rigorous methods of investigation and demonstration. This phase of analysis - rigor - is that which chiefly distinguishes the mathematics of the 19th from that of the 18th century. Before that time many writers employed more or less rigorous lines of reasoning but this was not generally true. Likewise since the time of Abel and Cauchy many writers have been lax, and this is even true of some of the work of these men themselves, but the tendency has been decidedly in favor of rigorous methods since the position taken by them in favor of such methods.

In some fields of mathematics rigor is not demanded. In drawing conclusions in applied mathematics we often need to come only within that degree of accuracy which is justified by the observed data. For example, the empirical formulas of mathematical physics often depend upon the

rapid convergence of the first few terms of infinite series. During the period of discovery it seems probably advisable that too strict regard for rigor be not made to hamper the worker, who should, however, apply strictly rigorous reasoning to the demonstration of the truth of the conclusions thus drawn.

2. The preceding remarks apply quite directly to the theory of divergent series. Previous to the time of Abel and Cauchy, mathematicians generally made use of such series as naturally presented themselves in the course of their investigations, usually without considering the question of their convergence or divergence. In spite of some hesitation at times they had very excellent reasons for considerable confidence in divergent series, even though they were numerical. The law Low, particularly with divergent series which were functions of a variable, the results of their calculations agreed with the analytical facts and afterwards with observations. Their conclusions were therefore sufficiently exact.

To be sure purely numerical divergent series may never be used for an exact calculation. There are, however, certain types of such series which may be used to obtain an approximate result. These are series whose terms decrease more or less rapidly in value for some number of terms, then begin to increase

and continue to do so to the end of the series. It frequently happen that a sufficiently close approximation for practical purposes can be obtained by calculating the value of these series up to the point where the terms begin to increase in value. In fact this approximation may in a given instance be closer than is required by the conditions of the problem especially if the value of the least term is quite small. To the applied mathematician such series are convergent series, especially so if the first terms decrease quite rapidly in value.

3. Thus there have come to be two quite distinct points of view of what constitutes a convergent or a divergent series. From the view-point of mathematical rigor, the pure mathematician is concerned with the ultimate character of the series without regard to what happens in the first thousand or so terms. From the view-point of practical application, the applied mathematician, as the physicist or astronomer, is, on the other hand, concerned almost entirely with the character of the earlier terms of the series. If the terms decrease in value for, say, the first 50 terms, even though they then begin and continue to increase in value, the series is to them a convergent series, of greater or less advantage from the practical standpoint according as the apparent

convergence is more or less rapid. These two points of view are not inconsistent but, in a way, they may rather be said to supplement each other. Practical progress would often be retarded were the applied mathematicians not to make more or less frequent use of such series. On the other hand were mathematical rigor to be lost sight of entirely, the false conclusions drawn would ultimately cause confusion and strife, and mathematics would soon cease to be the exact science it now is.

While Abel and Cauchy stood in general for rigorous methods of analysis, it was not without some hesitation that they entirely barred divergent series from consideration. In a letter* written shortly before his death, Abel remarked that it was a shame to attempt to establish any demonstration for divergent series; that it was a really extraordinary fact that results obtained from their use appeared in certain instances to justify that use, and that he was at that time engaged in an investigation of the reason for it and was finding the problem an interesting one.

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*Œuvres complètes d'Abel, édition Sylow-Lie, t. II, pp. 256-257. Abel said "les séries divergentes sont, en général, quelque chose de bien fatal, et c'est une honte qu'on ose y fonder aucune démonstration... la partie la plus essentielle des Mathematiques

At about the same time Cauchy sought to determine how the use of divergent series, while not at all theoretically legitimate, might lead in a very nearly certain manner to exact results. In the preface to his Analyse algébrique, 1821, Cauchy wrote: "I have been compelled to admit various propositions which may perhaps appear a little harsh; for example, that a divergent series has not a sum..." Cauchy never entirely lost sight of this problem and frequently refers to it in his works. In his memoir on the series of Stirling* he justified the process of approximate calculation for those series whose terms decrease in value for many terms, and then increase in value to the end of the series. The theory of Cauchy is moreover applicable to a much more general class of series known as asymptotic series, of which however Stirlings series was the only classical example up to that time.

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est sans fondement. Pour la plus grande partie les résultats sont justes, il est vrai, mais c'est là une chose bien étrange. Je m'occupe à en chercher la raison, problème très intéressant."

*Comptes Rendus, t. XVIII, p. 370. – Œuvres de Cauchy, série I, t. VIII, p. 18. This series was among the first divergent series to be practically employed. It served to calculate the function $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ when z is very large. This series will be treated later (see § 15).

Poincaré developed a general theory of this class of series, the general properties of which will be later explained.

5. Cauchy died in 1857 and the mathematicians of the next few decades seem to have taken from his statement quoted above the sentence, "A divergent series has not a sum", without the accompanying explanatory comments, and thus to have lost almost entirely his point of view. He really sought to determine under what conditions divergent series might be used, or to remove any uncertainty and justify the straight-forward statement his successors attributed to him.

After his death practically no work was done on divergent series until about 1880 when Laguerre published a memoire* on the integral $\int_x^\infty \frac{e^{-x}}{x} dx$. Since his memoire included nothing which did not directly concern this integral, it could not be made the basis of a general theory.

6. In 1886 appeared almost simultaneously two similar memoires, at once general and rigorous, one by Stieltjes,† the other by Poincaré.‡ Stieltjes considered a

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*Euvres de Laguerre, t. I, p. 428. — Bulletin de la Société mathématique de France, t. VII.

†Stieltjes, Recherches sur quelques séries semi-convergentes. (Annales de l'École normale, 1886).

‡Poincaré, Sur les intégrales irrégulières des équations linéaires. (Acta Mathematica, vol. VIII, 1886).

number of special examples and formed what he called semi-convergent series. The series treated by Poincaré he called asymptotic series and, as the term "semi-convergent" was a somewhat ambiguous one, having another generally accepted significance, the term introduced by Poincaré has been accepted.

Poincaré's asymptotic series is a divergent series of negative integral powers of a variable,

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \cdots,$$

to which a given function $F(x)$ has, by some process, given rise, which series is such that, if S_n represent the sum of the first $n+1$ terms, the condition

$$\lim_{x=\infty} x^n \left| F(x) - S_n \right| = 0$$

is satisfied for all values of n.

Calculations of the function made from the series are valid for large values of the variable. The series is such that, though ultimately divergent, yet for large values of x its terms at first decrease rapidly, then increase to the end of the series. The series is said to represent the function asymptotically.

7. Poincaré was led to a study of these series in connection with the preparation of his treatises on Celestial Mechanics. The solution of differential equations used by astronomers had given rise to series which formally

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*Poincaré, Méthodes nouvelles de la Mécanique céleste. t. II, III.

satisfied the equations and the astronomers had found that conclusions based on calculations made from these series were justified by their later observations. They had not concerned themselves with the ultimate convergence or divergence of the series and although Poincaré pointed out to them that they had been making use of divergent series, they continued to use them with good results. Poincaré then took up a consideration of the general problem of operations with such series with the result that he developed what may be called the first attempt at a general theory of divergent series. As will be seen in a later chapter, the simple operations of arithmetic hold good when applied to such series and this has made his work of very considerable importance.

To be sure the theory as developed by Poincaré has its drawbacks. To determine the asymptotic series corresponding to a given function — we call this series the asymptotic expansion of the function —, it is necessary to consider the limit, when x becomes infinite, of the product of x^n by the remainder obtained by subtracting S_n from $F(x)$. It is often very difficult to obtain a suitable expression for this remainder.

8. The excellent work of Poincaré has stimulated thought along the line of an extension to a more complete and general theory of divergent series.

E. Borel, E. W. Barnes and others have treated the con-

verse problem more or less extensively of determining the function which has given rise to a given asymptotic series, that is, of determining what may, for convenience, be called the "sum" of the divergent series.

Of the writers who have taken up a consideration of the asymptotic expansions of certain definite classes of functions, E.W. Barnes has done most in more recent years. He has studied transcendental integral functions of various types in connection with their lines of zeros and their asymptotic expansions for those points in the neighborhood of their essentially singular point $z = \infty$ which are not at a finite distance from any of the zeros of the functions considered. He hopes thus to be able to catalogue a very large number of classes of integral functions.

The applications of asymptotic series in the solutions of differential equations have been treated by Poincaré, Borel and others.

9. This thesis proper will be devoted to a consideration of the general properties of asymptotic expansions of given functions. The methods of obtaining the asymptotic expansions of given functions will be illustrated by examples. The converse problem of obtaining the function from the given asymptotic expansion, that is, of obtaining the "sum" of the divergent series, I shall not consider. I shall discuss

the fundamental operations and present some additional theorem which I have established. Under the not-always-permissible operation of differentiating an asymptotic series, I shall consider examples of series for which this operation is not permissible, and shall point out some classes of functions the derivative of whose asymptotic expansions are the asymptotic expansions of the derived functions.

As an illustration of the advantage which may attach to the use of a divergent asymptotic series instead of a convergent series in the calculation of the numerical value of a given function which has given rise to both series expansions, I shall consider the calculation of the value of the Bessel function $J_n(x)$, for $n = 0$ and $x = 10$, from two series, one being ultimately convergent, the other one ultimately divergent. Though the divergent asymptotic expansion of $J_n(x)$ will be developed and defined as valid for large values of x, yet for any value of x greater than 10, the first three terms of this series will be found to give the value of $J_0(x)$ or $J_1(x)$ correct to at least 6 places of decimals; on the other hand, for $x = 10$ it is necessary to use at least 20 terms of the convergent series to secure so good an approximation for $J_0(x)$.

As an illustration of the method, developed principally by E. Le Roy, of determining (what he has termed) the "asymptotic value" of certain given functions in the neighborhood of a singular point, for which value of the variable the series which represents the function becomes divergent, I shall discuss, with some detail, the first portion of an extremely interesting article by Le Roy* on "Valeurs asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réele".

I have made very free use of such material as has come to my notice and has suited my purpose, without, in many cases, giving specific credit to the author. Thereby acknowledge my indebtedness for material used and suggestions received in all such cases

*Le Roy, E., Bulletin des Sciences mathematiques, Série II, t. XXIV, 1900. [245-268.

Chapter I

Definitions And Illustrations.

10. Taylor's Expansion. — As a suitable method of leading up to an explanation of the asymptotic expansion of a function, let us consider the representation of a function by means of a Taylor's series. Take an interval $(0,\alpha)$ of the positive real axis, and let $f(x)$ denote any real function which is continuous and has $n+1$ successive derivatives at every point within the interval. We make no hypothesis concerning the character of the function at the extremities of the interval except to suppose that

$$\lim_{x=0+} \frac{f^{(n)}(x)}{n!} = a_n \qquad (n=0,1,2,\cdots,n).$$

Thus the function is represented at any point within the interval $(0,\alpha)$ by the Taylor's series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x) \quad (0<\theta<1).$$

If $f(x)$ has an unlimited number of successive derivatives and if further $\lim_{x=0+} \frac{f^{(n)}(x)}{n!}$ exists and equals a_n for all values of n, the number of terms in the series can be increased to any desired number. The function thus gives rise to a series

$$a_0 + a_1 x + a_2 x^2 + \cdots, \tag{1}$$

which is uniquely determined for any point within

the interval $(0, \alpha)$ by the limiting values of the function and of its derivatives.

On the other hand a given series does not uniquely determine the function for many functions can give rise in the same interval, having the origin for one end point, to the same Taylor's series. Thus the exponential functions of the form e^{-1/x^r} $(r = 1, 2, 3, \cdots)$ possess the properties initially defined and have a Taylor's expansion identically zero for all points in the interval $(0, \alpha)$. Consequently the functions $f(x)$ and $\varphi(x) = f(x) + k e^{-1/x^r}$ have identically the same expansion in Taylor's series.

Since the correspondence between the function and the series is not reversibly unique, the series can not, in general, be used to determine the generating function. Nevertheless it is important to notice that we may make use of the series to calculate the value of the function with a very considerable degree of accuracy. Let S_n represent the sum of the first $n+1$ terms, n being some fixed positive integer. By sufficiently diminishing the value of x each term of the series can be made as small as we please in comparison with the next preceding term, thus causing the series at the beginning to have the appearance of converging rapidly, although it may ultimately diverge. As the value of x continues to decrease this apparent convergence exists for an increasing number of terms, if not throughout its entire extent. Since by hypothesis the function $f(x)$ was unkn-

itedly differentiable within the interval $(0,\alpha)$, therefor the successive derivatives are continuous within this interval. Moreover by hypothesis,

$$\lim_{x=0+} \frac{f^{(n+1)}(x)}{(n+1)!} = a_n,$$

and consequently if we contract the interval $(0,\alpha)$ sufficiently, we can make

$$\left| a_{n+1} - \frac{f^{(n+1)}(x)}{(n+1)!} \right|$$

as small as we please. We then have for the remainder in Taylor's series

$$R_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!} \cdot x^{n+1} = a_{n+1} x^{n+1}(1+\delta) \qquad (|\delta| < \epsilon), \tag{2}$$

where ϵ is an arbitrarily small positive number. This shows that for sufficiently small values of x, that is, for a sufficiently contracted interval $(0,\alpha)$, the sum of the first $n+1$ terms of the series (1) represents the generating function with an error approximately equal to the first term neglected.

In making this calculation there is nothing to indicate exactly when x is sufficiently small to give the required degree of approximation. If the value obtained from the series for a given value of x holds for a large number of consecutive values of n, then the best approximation to the value of the function consistent with our information evidently would be obtained by the use of all the terms of the series up to and including the term of least absolute value. The error is then of the same order of magnitude as is the last term used or the first term neglected.

It is at once apparent that equation (2) both gives the limit to the error made in stopping with the $(n+1)$th term and shows that this limit grows smaller in absolute value as x diminishes. That is to say, as x diminishes, a greater and greater number of terms must be used to give the best approximation to the value of the generating function. These facts may be expressed by saying that the approximation given by the series is of an asymptotic character. This holds true whether the series is ultimately convergent or divergent.

11. Definitions of Asymptotic Expansions.—I. For small values of the variable. Now let us suppose that a given function $f(x)$ has given rise to the divergent series

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

If then S_n represents the sum of the first $n+1$ terms of this series and

$$\lim_{x=0+} \left| x^{-n} \left(f(x) - S_n \right) \right| = 0, \qquad (n = 0, 1, 2, 3, \cdots),$$

we say that the series represents the function $f(x)$ asymptotically for very small values of the variable x. This is the equivalent of the definition adopted by Poincaré* in his development of the arithmetic theory of asymptotic convergence, which will be treated below.

It will be noticed that no hypothesis has been made here concerning the nature of the generating function. It is not assumed that the function can be unlimitedly

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*Poincaré, Acta Mathematica, vol. 8, 1886, pp. 295-344

differentiated within the interval of definition. We are therefore no longer concerned with the question of convergency and in fact the theory of asymptotic series has been developed with the especial idea of taking care of those cases in which the series are divergent. On account of the absence of these restrictions which were vitally essential to the Taylor's expansion, the asymptotic representation of functions may be said to cover a broader field than does the representation of functions by Taylor's expansion. The application of the theory of asymptotic representation is also thereby considerably simplified.

II. For large values of the variable. Again let us suppose that a given function $f(x)$ has given rise to a divergent series of the form

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \cdots .$$

If we let S_n represent the sum of the first $(n+1)$ terms, then if

$$\lim_{x=\infty} \left| x^n (f(x) - S_n) \right| = 0, \quad (n = 0, 1, 2, 3, \cdots),$$

we say that the series represents the function asymptotically for very large values of the variable x. This is the definition given by Poincaré.

In each of the above definitions* it is assumed for the sake of simplicity that the values of x are taken along the positive real axis. In some instances the expansion represents the function asymptotically for values of x taken along some other line through the origin and having a fixed direction. In still other cases it

*We shall in this thesis consider asymptotic expansions for real values only.

series may be shown to represent the function asymptotically for values of the variable determined by points lying within a circular sector whose apex is the origin and bounding radii make known definite angles with the real axis.

That these two definitions are consistent, the first defining asymptotic representation, or asymptotic equivalence, for points along the positive real axis near the origin, the second for very large positive real values of the variable, may be shown as follows:

Let $f(x)$ be represented asymptotically by

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \cdots + \frac{A_n}{x^n} + \cdots$$

for very large positive real values of the variable x.

Then, by definition,

$$\lim_{x=\infty} \left| x^n \left(f(x) - S_n(x) \right) \right| = 0.$$

Now let $x = \frac{1}{t}$, so that $t = \frac{1}{x}$; then $f\left(\frac{1}{t}\right)$, which we shall call $F(t)$, is represented asymptotically by

$$A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n + \cdots$$

for very large positive real values of x, that is, for very small positive real values of t.

For we have

$$\lim_{x=\frac{1}{t}=\infty} \left| x^n \left(f(x) - S_n(x) \right) \right| = \lim_{t=0+} \left| \frac{1}{t^n} \left(f\left(\frac{1}{t}\right) - S_n\left(\frac{1}{t}\right) \right) \right|$$

$$= \lim_{t=0+} \left| t^{-n} \left(F(t) - S_n'(t) \right) \right| = 0.$$

Therefore the t-series represents $F(t)$ asymptotically for points on the positive real t-axis near the origin.

III. For points near $x = a$. More generally we may say that

the series

$$A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots$$

represents the function $f(x)$ asymptotically for real values of x in the neighborhood of $x=a$ if

$$\lim_{x \doteq a+} \left| (x-a)^{-n} \{ f(x) - S_n(x-a) \} \right| = 0.$$

We prove this as follows.

Let $F(t)$ be represented asymptotically by the series

$$A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n + \cdots$$

for positive real values of t near the origin. Then we have

$$\lim_{t \doteq 0+} \left| t^{-n} \{ F(t) - S_n(t) \} \right| = 0.$$

Now let $t = x - a$; then when $t \doteq 0+$, $x - a \doteq 0+$, and $x \doteq a+$, and we have

$$\lim_{t = x-a \doteq 0+} \left| t^{-n} \{ F(t) - S_n(t) \} \right| = \lim_{x-a \doteq 0+} \left| (x-a)^{-n} \{ F(x-a) - S_n(x-a) \} \right|$$

$$= \lim_{x \doteq a+} \left| (x-a)^{-n} \{ F(x-a) - S_n(x-a) \} \right|$$

$$= \lim_{x \doteq a+} \left| (x-a)^{-n} \{ f(x) - S_n(x-a) \} \right| = 0,$$

and therefore $F(t) = F(x-a) = f(x)$ is represented asymptotically by the series.

$$A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots.$$

Q.E.D.

12. The Error involved in the use of an Asymptotic Expansion. - If the function $J(x)$ is represented asymptotically by

$$A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \cdots,$$

then it is evident that, although the series be ultimately divergent, yet it is possible to take x sufficiently large that we shall have for an assigned value of n,

$$x^n (J - S_n) < \epsilon,$$

where ϵ is an arbitrarily small positive number. The error,

$$J - S_n = \frac{\epsilon}{x^n},$$

which is made in the value of $J(x)$ by taking the sum of the first $n+1$ terms of the series is then extremely small. Moreover, it is very much smaller than that made by taking only n terms and which is equal to

$$J - S_{n-1} = \frac{A_n + \epsilon}{x^n},$$

wherein ϵ is in general infinitely small as compared with $A_n + \epsilon$.

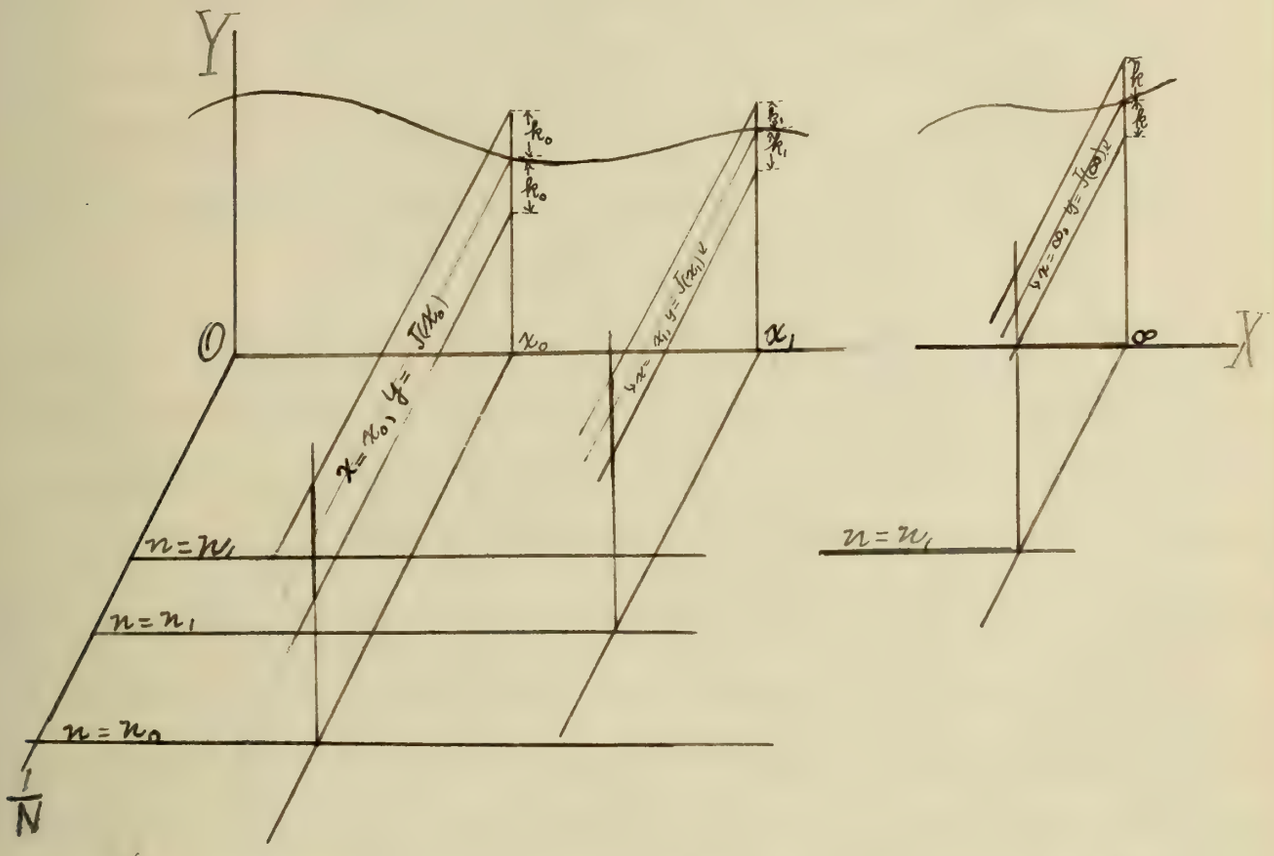
From this we see that, if x is very large, the terms of the series decrease very rapidly at first and then begin at some term to increase and continue to increase to the end of the series, and that, in spite of its ultimate divergence, it will be legitimate to make use of the series for the approximate calculation of the value of J for large values of the variable.

13. Geometric Treatment of Asymptotic Approximation.- Let the series

$$S(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \cdots$$

represent the given function $J(x)$ asymptotically for large positive real values of x. Let us consider the curve $y = J(x)$ in connection with the curves $y = S_n(x)$, $(n = 1, 2, 3, \cdots)$, $S_n(x)$ being the sum of the first $n+1$ terms of the series $S(x)$. Let us draw the curve $y = J(x)$ in the X-Y plane, and the curves

$y = S_1(x), S_2(x), \cdots, S_n(x), \cdots$ in the planes whose equations are $n = 1, 2, 3, \cdots, n, \cdots$, the values of n being platted along the $\frac{1}{n}$ axis which is perpendicular to the X-Y plane through the origin.



The successive curves $y = S_1(x)$, $y = S_2(x)$, $\cdots$, $y = S_n(x)$, $\cdots$ will cut every plane parallel to the Y-$\frac{1}{N}$ plane in a series of points which we may designate for the planes $x = x_0, x = x_1, x = x_2, \cdots, x = x_m, \cdots$ by

$x = x_0$, $(x_0, \frac{1}{1}, y_{1,0}), (x_0, \frac{1}{2}, y_{2,0}), (x_0, \frac{1}{3}, y_{3,0}), \cdots, (x_0, \frac{1}{n_0}, y_{n_0,0}), \cdots$

$x = x_1$, $(x_1, \frac{1}{1}, y_{1,1}), (x_1, \frac{1}{2}, y_{2,1}), (x_1, \frac{1}{3}, y_{3,1}), \cdots, (x_1, \frac{1}{n_1}, y_{n_1,1}), \cdots$

$x = x_2$, $(x_2, \frac{1}{1}, y_{1,2}), (x_2, \frac{1}{2}, y_{2,2}), (x_2, \frac{1}{3}, y_{3,2}), \cdots, (x_2, \frac{1}{n_2}, y_{n_2,2}), \cdots$

. . . . . .

$x = x_m$, $(x_m, \frac{1}{1}, y_{1,m}), (x_m, \frac{1}{2}, y_{2,m}), (x_m, \frac{1}{3}, y_{3,m}), \cdots, (x_m, \frac{1}{n_m}, y_{n_m,m}), \cdots$

. . . . . .

Let the $(n_0+1)^{\text{th}}$ be the term of least absolute value in the series

$$S(x_0) = c_0 + \frac{c_1}{x_0} + \frac{c_2}{x_0^2} + \frac{c_3}{x_0^3} + \cdots + \frac{c_{n_0}}{x_0^{n_0}} + \cdots,$$

it being understood that this series converges rapidly up to and including the (n_0+1)th term. The successive points indicated above as lying in the plane $x = x_0$ will then for $n \leqq n_0$ lie outside of a rectangle of width $2k_0$ lying in the plane $x = x_0$ and having for its median the line $x = x_0$, $y = J(x_0)$, where

$$k = |J(x_0) - S_{n_0}(x_0)|,$$

and k_0 may be said to be a measure of the approximation of the series $S(x_0)$ to the value of the function $J(x_0)$, and the successive curves approximation curves to the curve $y = J(x)$.

Now since the coefficients in the series $S(x)$ are independent of x, then there will be some value of $x > x_0$, say $x = x_1$, for which the terms of the series will continue to decrease for more than n_0+1 terms, say for n_1+1 terms, and then increase throughout the remaining terms of the series. $S_{n_1}(x_1)$ will give the best approximation to the value of the function $J(x_1)$, and if we let

$$k_1 = |J(x_1) - S_{n_1}(x_1)|,$$

then $k_1 \leqq k_0$, that is the approximation will be at least as good as that previously given. It will be possible to take x_1 sufficiently large that $k_1 < k$, which we suppose to be the case. Then the series of points $(x_1, \frac{1}{1}, y_{1,1})$, $(x_1, \frac{1}{2}, y_{2,1})$, $\cdots$, in which the successive approximation curves cut the plane $x = x_1$ will for $n \leqq n_1$ lie outside of a rectangle of width $2k_1$ in the plane $x = x_1$ and having for its median the line $x = x_1$, $y = J(x_1)$.

Now since $J(x)$ is represented asymptotically by the series

$$c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \cdots$$

for large values of x, we have

$$\lim_{x=\infty} x^n \left| J(x) - S_n(x) \right| = 0$$

for all values of n up to and including n, then if we place

$$k = \left| J(x) - S_n(x) \right|,$$

we shall have

$$\lim_{x=\infty} x^n \left| J(x) - S_n(x) \right| = \lim_{x=\infty} x^n \cdot k = 0,$$

that is, k is a zero of at least the $(n+1)$th order.

From this we see that the successive approximation curves $y = S_1(x)$, $y = S_2(x)$, $y = S_3(x)$, $\ldots$, $y = S_n(x)$, will cut the plane $x = \infty$ in a series of points which will for $n \leqslant n_1$ lie without a rectangle of width $2k$ in the plane $x = \infty$ having for its median the line $x = \infty$, $y = J(\infty)$.

If then we imagine the curves $y = S_1(x)$, $y = S_2(x)$, $y = S_3(x)$, $\ldots$, $y = S_n(x)$, projected on the X-Y plane, we see that they will approach the curve $y = J(x)$ at the infinitely distant point on this curve and that the curves $y = S_{n+1}(x)$, $y = S_{n+2}(x)$, $\ldots$, will begin to recede from the curve $y = J(x)$, whenever x has that value which causes the first $n+1$ terms to converge, the $(n+1)$th term to be the term of least absolute value, and the terms to then begin to increase in value and to continue to do so throughout the remaining terms of the series. Thus the curves may be said to approach the curve $y = J(x)$ asymptotically.

Now although we have shown that these approximation curves approach the curve $y = J(x)$ for very large

ues of x, it will often happen that for comparatively small values of x, say $x = 10$ or 100, the approximation given by the series when the sum of the terms out to and including the term of least absolute value is taken, will be sufficiently good for practical purposes and much better than the conditions of the problem at hand require. In many cases therefore we may calculate with a sufficient degree of approximation the value of the function for relatively small values of the variable by using a much smaller number of terms than are required to give the best approximation, that is, we need not continue the calculation to the term of least absolute value. Geometrically, we need consider fewer of the approximation curves above referred to than are necessary to make any particular k take the smallest value possible.

14. Illustration of Asymptotic Expansion. -- As a simple example of an asymptotic expansion let us consider the function $f(x)$ defined by the relation

$$f(x) = \int_x^\infty \frac{e^{x-t}}{t}\, dt,$$

where x is real and positive and the path of integration is the real axis in the t-plane.

Integrating the given function by parts and repeating the process with the remaining integral, we obtain

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + (-1)^n \frac{n!}{x^{n+1}} + (-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{x-t}}{t^{n+2}}\, dt$$

$$= S_n + (-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{x-t}}{t^{n+2}}\, dt.$$

In the series thus obtained, the ratio of the n^{th} to the $(n-1)$ th term is $-\frac{n-1}{x}$, which in absolute value is greater than unity for all values of n greater than $1+x$. The series S is therefore divergent for all values of x. However we shall now show that under certain circumstances the series can be used for the approximate calculation of the value of $f(x)$ for a definite value of x.

We have

$$|f(x) - S_n| = (n+1)! \int_x^\infty \frac{e^{x-t}}{t^{n+2}}\, dt$$

$$< (n+1)! \int_x^\infty \frac{dt}{t^{n+2}}, \quad \because e^{x-t} < 1, \text{ and } t > 0,$$

$$< \frac{n!}{x^{n+1}}.$$

Therefore $$x^n |f(x) - S_n| < \frac{n!}{x}.$$

If now we select some finite value of n and calculate S_n we have for very large values of x

$$\lim_{x=\infty} x^n |f(x) - S_n| < \lim_{x=\infty} \frac{n!}{x}$$

and, since the second member of this inequality is zero, we have

$$\lim_{x=\infty} x^n |f(x) - S_n| = 0;$$

whence

$$f(x) \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + (-1)^n \frac{n!}{x^{n+1}} + \cdots.^*$$

Thus we see that the value of the function $f(x) = \int_x^\infty \frac{e^{x-t}}{t}\, dt$

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* Read "$f(x)$ is asymptotically equivalent to $\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots$," or "the function $f(x)$ is asymptotically represented by the series $\frac{1}{x} - \frac{1}{x^2} + \cdots$."

can be calculated with great accuracy for large values of x, by taking the sum of a finite number of terms of the infinite divergent series whose generating function is $f(x)$. For any particular value of x, we ought to use for this calculation the terms up to and including the term of least absolute value to secure the best approximation to the value of the generating function for that particular value of x.

15. Asymptotic Expansion of Log $\Gamma(z)$. Stirling's Series.- As another illustration let us now find the asymptotic expansion of the function $\log \Gamma(z)$, which expansion is used in computing the numerical value of $\Gamma(z)$.

The function $\Gamma(z)$ may be defined by the integral

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

the path of integration being the positive real axis. This integral is called the Eulerian Integral of the second kind. There is some objection to defining the Gamma-function in this manner, since the integral has existence only when the real part of z is positive. For simplicity, we shall use only positive real values of z.*

Integrating by parts and simplifying by comparison, we have

$$\Gamma(z+1) = z\,\Gamma(z).$$

This is one of the most characteristic properties of the

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*For a proof and discussion of the expansion when z has complex values, see an article by Stieltjes, Liouville's Jour., Vol. 4, 1889, pp. 425-444.

Gamma-function. When n is a positive integer we have from it

$$\Gamma(n+1) = n!$$

When n is very large the calculation of the value of $\Gamma(n+1)$ becomes very tedious and troublesome and it is therefore important, especially in many questions that arise in the calculation of probabilities, to know a value approached by it.

The expansion from which this calculation is made is an expansion for $\log_e \Gamma(z+1)$. From the relation expressed above we see that

$$\log \Gamma(z+1) = \log z + \log \Gamma(z),$$

and that $\log \Gamma(z)$ is thus closely related to $\log \Gamma(z+1)$.

Binet* gave the following expansion for $\log \Gamma(z)$,

$$\log \Gamma(z) = (z-\tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi) + 2\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{z}\right)}{e^{2\pi x}-1}\, dx,$$

whence we obtain at once

$$\log \Gamma(z+1) = (z+\tfrac{1}{2}) \log z + \tfrac{1}{2} \log(2\pi) + 2\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{z}\right)}{e^{2\pi x}-1}\, dx.$$

This expansion is of great importance as showing the way in which $\log \Gamma(z)$ increases as z becomes very large; as z increases indefinitely, the last integral diminishes indefinitely, and therefore the remaining terms furnish an approximate expression for $\log \Gamma(z+1)$ when z is large.

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* Binet, Journal de l'École Polyt., XVI, 1839, pp 123-343. For a discussion of Binet's development see Whittaker, Modern Analysis, 1902, pp. 186-189. For another expression for $\log \Gamma(z)$, see Jordan, Cours d'Analyse, 2d edition, vol. II, p 180.

Let us write

$$\log \Gamma(z+1) = (z+\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \omega(z),$$

where

$$\omega(z) = 2\int_0^\infty \frac{\tan^{-1}\left(\frac{x}{z}\right)}{e^{2\pi x}-1}\,dx.$$

Now

$$\tan^{-1}\left(\frac{x}{z}\right) = \frac{x}{z} - \frac{1}{3}\frac{x^3}{z^3} + \frac{1}{5}\frac{x^5}{z^5} - \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)z^{2n-1}} + \frac{(-1)^n}{z^{2n-1}}\int_0^x \frac{u^{2n}}{u^2+z^2}\,du,$$

as can be easily verified.

We have also

$$\int_0^\infty \frac{x^{2n-1}}{e^{2\pi x}-1}\,dx = \frac{B_n}{4n};^*$$

where

$$B_n = 2\frac{(2n)!}{(2\pi)^{2n}}\left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \cdots\right).$$

The numbers thus defined are the numbers of Bernoulli.†

Making use of these relations in the expression for $\omega(z)$, we have on simplifying

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* For proof, see Carda, Monatshefte für Math. und Phys., vol. V, 1894, pp. 321–324. The proof is there made to depend upon contour integrals.

† For a discussion of Bernoullian numbers see the following:

(a) Carr, Synopsis of Mathematics, p 276; also formulae 1539, 1543, 1544, 1545 for equations involving the B's and the calculations of B_n, $n=1,2,\cdots,10$.

(b) Chrystal's Algebra, vol. II, p. 207.

(c) Whittaker, Modern Analysis, 1902, p 97, §63 on Bernoullian Nos. & Polyn.

(d) Glaisher, J. W. L., "Fundamental theorems relating to Bernoullian numbers," Messenger of Mathematics, vol. XXIX, 1899, pp. 49–63.

$$\omega(z) = \frac{B_1}{1\cdot 2}\frac{1}{z} - \frac{B_2}{3\cdot 4}\frac{1}{z^3} + \frac{B_3}{5\cdot 6}\frac{1}{z^5} - \cdots + (-1)^{n-1}\frac{B_n}{(2n-1)\,2n}\frac{1}{z^{2n-1}} + (-1)^n \frac{2}{z^{2n-1}}\int_0^\infty \frac{dx}{e^{2\pi x}-1}\int_0^x \frac{u^{2n}du}{u^2+z^2}$$

$$= \sum_{r=1}^{n} \frac{(-1)^{r-1}B_r}{2r(2r-1)}\frac{1}{z^{2r-1}} + (-1)^n \frac{2}{z^{2n-1}}\int_0^\infty \frac{dx}{e^{2\pi x}-1}\int_0^x \frac{u^{2n}du}{u^2+z^2}.$$

That this development for $\omega(z)$ is divergent may be readily seen. The absolute value of the ratio of the $(p+1)$th to the pth term of the series is

$$\frac{B_{p+1}}{B_p}\frac{(2p-1)(2p)}{(2p+1)(2p+2)}\frac{1}{z^2}.$$

now
$$\frac{B_{p+1}}{B_p} = \frac{(2p+1)(2p+2)}{4\pi^2}\frac{\left(1+\frac{1}{2^{2p+2}}+\frac{1}{3^{2p+2}}+\cdots\right)}{\left(1+\frac{1}{2^{2p}}+\frac{1}{3^{2p}}+\cdots\right)};$$

whence
$$\frac{B_{p+1}}{B_p} > \frac{(2p+1)(2p+2)}{4\pi^2}\frac{1}{1+\frac{1}{2^{2p}}+\frac{1}{3^{2p}}+\cdots} > A(2p+1)(2p+2),$$

where A is a proper positive fraction.

Therefore the above expression for the ratio increases indefinitely with p whatever be the value of z.

We have made the calculation of $\log \Gamma(z+1)$ depend upon the calculation of $\omega(z)$ and now we have reduced this to the calculation of the last term of the above expression for $\omega(z)$, the approximate magnitude of which we shall now find when z is very large.

The quantity
$$\int_0^\infty \frac{dx}{e^{2\pi x}-1}\int_0^x \frac{u^{2n}du}{u^2+z^2}$$
is less than
$$\frac{1}{z^2}\int_0^\infty \frac{dx}{e^{2\pi x}-1}\int_0^x u^{2n}du,$$
which is equal to

$$\frac{1}{(2n+1)z^2}\int_0^\infty \frac{x^{2n+1}dx}{e^{2\pi x}-1},$$

that is, to

$$\frac{B_{n+1}}{4(n+1)(2n+1)z^2};$$

whence we have

$$\int_0^\infty \frac{dx}{e^{2\pi x}-1}\int_0^x \frac{u^{2n}du}{u^2+z^2} < \frac{B_{n+1}}{4(n+1)(2n+1)z^2}.$$

Therefore we may write $\omega(z)$ as follows

$$\omega(z) = \frac{B_1}{1\cdot 2}\frac{1}{z} - \frac{B_2}{3\cdot 4}\frac{1}{z^3} + \frac{B_3}{5\cdot 6}\frac{1}{z^5} - \cdots + (-1)^{n-1}\frac{B_n}{(2n-1)\,2n}\frac{1}{z^{2n-1}}$$
$$+\theta(-1)^n\frac{z}{z^{2n-1}}\cdot\frac{B_{n+1}}{4(n+1)(2n+1)z^2}$$
$$= \sum_{r=1}^{n}\frac{(-1)^{r-1}B_r}{2r(2r-1)}\frac{1}{z^{2r-1}} + \theta(-1)^n\frac{B_{n+1}}{(2n+1)(2n+2)}\frac{1}{z^{2n+1}} \quad (0<\theta<1).$$

If we now choose a definite value of n, the last term of this expansion can be made as small as we please by taking z sufficiently large. It follows at once that, if S_{n-1} represents the sum of the first n terms of this series,

$$\lim_{z=\infty} z^{2n-1}\left|\omega(z) - S_{n-1}\right| = 0,$$

and therefore we have the asymptotic equality

$$\omega(z) \sim \frac{B_1}{1\cdot 2}\frac{1}{z} - \frac{B_3}{3\cdot 4}\frac{1}{z^3} + \frac{B_3}{5\cdot 6}\frac{1}{z^5} - \cdots,$$

that is, this series is the asymptotic expansion of the function $\omega(z)$ for large real positive values of z.

Returning now to the function $\log\Gamma(z+1)$ we have the asymptotic equality

$$\log\Gamma(z+1) \sim (z+\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_n}{2n(2n-1)}\frac{1}{z^{2n-1}};$$

that is, this series is the asymptotic expansion of the function $\log\Gamma(z+1)$ for large real positive values of z.

This is generally known as Stirling's Series. It is among the first, if not the first, divergent series of which practical use was made and was employed in the calculation of $n!$ for large positive integral values of n.

On account of the relation between $\log \Gamma(z+1)$ and $\log \Gamma(z)$, we have at once the asymptotic equality

$$\log \Gamma(z) \sim (z-\tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{2n(2n-1)} \frac{1}{z^{2n-1}}.$$

For large values of z the terms of this series decrease in value at first very rapidly, then less rapidly, and then they begin to increase in value and continue to increase throughout the remaining terms of the series. To obtain the best approximation to $\log \Gamma(z+1)$ or $\log \Gamma(z)$, it is only necessary to stop in the calculation with the term of least absolute value in the expansion for $\alpha(z)$, that is, with the term preceding that with which begins the increase in value of the terms. The approximation is then better than the first term neglected.

In a particular investigation it is not always necessary to carry the calculation out so as to secure the best approximation and frequently the first few terms of an expansion will furnish a sufficiently good approximation for the case under consideration.

16. Uniqueness of an Asymptotic Expansion. - From the fundamental definition of the asymptotic expansion of a given function, it follows at once that if a given function

has an asymptotic expansion that expansion is unique. We have heretofore remarked (see §11) that we have confined ourselves to a consideration of the asymptotic expansions of functions for positive real values of the variable, but that such expansions may be obtained which are valid for values of the variable taken along some other line through the origin than the positive real axis, or indeed for complex values of the variable determined by points lying within a circular sector having its apex at the origin and bounding radii making definite angles with the real axis.

The asymptotic expansion of the given function will be unique for values of the variable constrained to lie along a particular line or within a particular sector, and may be entirely different for values of the variable taken along some other line or within a different sector.

On the other hand, for the same argument of x, the same series may represent asymptotically many different functions. To show that this is true it is sufficient to show that there are functions $J(x)$ which are represented asymptotically by a series $\sum_{n=0}^{\infty} \frac{c_n}{x^n}$, wherein $c_n = 0$ for all values of n, that is, $J(x)$ is such that

$$\lim_{x=\infty} x^n J(x) = 0,$$

whatever n may be.

Consider $e^{-\alpha x^\beta}$, α and β being positive integers. We have

$$\lim_{x=\infty} e^{-\alpha x^\beta} = 0 = c_0, \text{ say.}$$

$$\lim_{x=\infty} x\left|e^{-\alpha x^\beta} - c_0\right| = 0 = c_1$$

$$\lim_{x=\infty} x^2 \left| \left(e^{-qx^\beta} - c_0 - \frac{c_1}{x} \right) \right| = 0 = c_2,$$

. . . . . .

$$\lim_{x=\infty} x^n \left| e^{-qx^\beta} - \left(c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} \right) \right| = 0 = c_n,$$

whatever be the value of n. From this we have by virtue of the fundamental definition,

$$e^{-qx^\beta} \sim \sum_{n=0}^{\infty} \frac{c_n}{x^n}, \text{ and } \sum_{n=0}^{\infty} \frac{c_n}{x^n} \equiv 0,$$

that is, e^{-qx^β} for all positive integral values of q and β has an asymptotic expansion which is identically zero. Then obviously the functions

$$J(x) \text{ and } J(x) + ke^{-qx^\beta}$$

will give rise to identically the same asymptotic series.

We are, however, frequently able to identify the particular function which has given rise to an asymptotic expansion by other conditions. For example, if we are considering a differential equation, the fact that the function sought is an integral of the equation will often be sufficient to enable us to uniquely determine which one of a class of functions an asymptotic expansion represents.*

17. <u>A Method of obtaining the Asymptotic Expansion of a given Function</u>.— If we are given any function we can determine whether or not it has an asymptotic expansion. Let the given function $J(x)$ be a function of positive

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* See Borel. É., Leçons sur les series divergentes, 1901, section on "Applications aux équations differentielles", pp 36 et seq.

real values of x. Let us suppose that when x is indefinitely increased, $J(x)$ has a limit

$$\lim_{x=\infty} J(x) = C_0.$$

We may say that C_0 gives an approximation to $J(x)$ for very large values of the variable. Let us now see if we can obtain a better approximation. We have

$$\lim_{x=\infty} (J(x) - C_0) = 0.$$

Now let us suppose that for x indefinitely large the product of this quantity by x has a limit

$$\lim_{x=\infty} x\,|J(x) - C_0| = C_1,$$

whence

$$\lim_{x=\infty} x\left|J(x) - \left(C_0 + \frac{C_1}{x}\right)\right| = 0,$$

and $C_0 + \frac{C_1}{x}$ may be said to give a better approximation to the function $J(x)$ for very large positive real values of x.

If now we have also

$$\lim_{x=\infty} x^2\left|J(x) - \left(C_0 + \frac{C_1}{x}\right)\right| = C_2,$$

then

$$\lim_{x=\infty} x^2\left|J(x) - \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2}\right)\right| = 0,$$

and $C_0 + \frac{C_1}{x} + \frac{C_2}{x^2}$ will give a still better approximation to the function $J(x)$ for very large positive real values of x.

Continuing this process let us suppose we have

$$J(x) = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x^3} + \cdots + \frac{C_n}{x^n} + \cdots,$$

the various coefficients having been obtained in the manner above indicated.

If the series thus obtained is a series which although ultimately divergent is, for large positive real values of x,

more or less rapidly convergent for the early terms, then the series is said to represent the function asymptotically, that is, the given function $J(x)$ is said to have an asymptotic expansion valid for large values of x.

Being given a function it is only necessary to apply this method therefore to determine whether or not the function has an asymptotic development. The resulting series is that development.

It will be clearly seen that this is in exact agreement with the definition of arithmetic asymptotic equivalence as set up by Poincaré. For if in the series obtained for $J(x)$, we place

$$S_n(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n},$$

then we have said that the necessary and sufficient condition that the given function $J(x)$ have the series as its asymptotic expansion is that

$$\lim_{x=\infty} x^n \left| J(x) - S_n(x) \right| = 0,$$

for all values of n.

Now for the purpose of numerical computation it is often unnecessary to secure the best, precisely approximation, which in the above case would be obtained for any particular large positive real value of x by computing the series for this value of x out to and including the term of least absolute value. The series obtained will usually converge very rapidly for large values of x, and it frequently happens that the first few terms, often as few as four terms, will

give an approximation for even comparatively small values of x which is much closer than the conditions of the problem at hand require or than the data on which the computations are based will justify. In a case of this kind, from the viewpoint of the computer it will often be unnecessary to determine the general term, since

$$c_0, \quad c_0 + \frac{c_1}{x}, \quad c_0 + \frac{c_1}{x} + \frac{c_2}{x^2}, \quad \ldots$$

as determined above, formed apart of approximations to the value of the given function for large values of the variable. Compare this with the "Geometric treatment of asymptotic approximation", treated above in §20.

That the above is not the only method of procedure for obtaining the asymptotic expansion of a given defined function is at once apparent. If the given function is defined by means of a definite integral, it will happen frequently that the continued integration by parts - or by some other method - will give an infinite series which, although ultimately divergent, may nevertheless be used to calculate the value of the given function for large values of the variable. An instance of this kind is the asymptotic expansion of the function defined by

$$f(x) = \int_x^\infty \frac{e^{x-t}}{t}\, dt,$$

given in §14.

We shall extend this article in the chapter on "Operations with asymptotic series".

Chapter II.

Operations With Asymptotic Series.

18. In this chapter we shall define the simple operations for asymptotic series and study the effect on such series of performing these operations on the functions which gave rise to them.

19. Equality of two Asymptotic Series. — The two asymptotic series $\sum_{n=0}^{\infty} \frac{c_n}{x^n}$ and $\sum_{n=0}^{\infty} \frac{c'_n}{x^n}$ are said to be equal when, and only when, $c_n = c'_n$ $(n = 0, 1, 2, 3, \ldots)$.

20. Addition of Asymptotic Series. —

Theorem. If $\sum_{n=0}^{\infty} \frac{c_n}{x^n}$ and $\sum_{n=0}^{\infty} \frac{c'_n}{x^n}$ are two series which represent asymptotically the functions $f(x)$ and $f'(x)$ respectively, then the series $\sum_{n=0}^{\infty} \frac{c_n + c'_n}{x^n}$ represents asymptotically the function $f(x) + f'(x)$.

Let S_n and S'_n be the sum of the first $n+1$ terms of the two given asymptotic series. Then we may write

$$f(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} + \frac{\epsilon_n}{x^n} = S_n + \frac{\epsilon_n}{x^n},$$

$$f'(x) = c'_0 + \frac{c'_1}{x} + \frac{c'_2}{x^2} + \cdots + \frac{c'_n}{x^n} + \frac{\epsilon'_n}{x^n} = S'_n + \frac{\epsilon'_n}{x^n},$$

where

$$\epsilon_n = \frac{c_{n+1}}{x} + \frac{c_{n+2}}{x^2} + \cdots, \quad \epsilon'_n = \frac{c'_{n+1}}{x} + \frac{c'_{n+2}}{x^2} + \cdots,$$

whence it follows that

$$\lim_{x=\infty} \epsilon_n = 0, \text{ and } \lim_{x=\infty} \epsilon'_n = 0.$$

Then we have

$$J(x) + J'(x) = (C_0 + C_0') + \frac{C_1 + C_1'}{x} + \frac{C_2 + C_2'}{x^2} + \cdots + \frac{C_n + C_n'}{x^n} + \frac{\epsilon_n + \epsilon_n'}{x^n}$$
$$= S_n + S_n' + \frac{\epsilon_n + \epsilon_n'}{x^n}.$$

Now $$\lim_{x=\infty} x^n \left| \{J(x) + J'(x)\} - \{S_n + S_n'\} \right| = \lim_{x=\infty} (\epsilon_n + \epsilon_n') = 0;$$

therefore the series $S_n + S_n' + \frac{\epsilon_n + \epsilon_n'}{x^n}$, that is, $\sum_{n=0}^{\infty} \frac{C_n + C_n'}{x^n}$, represents the function $J(x) + J'(x)$ asymptotically, which proves the theorem.

21. Corollary I. *The series which represents asymptotically the sum of any finite number of functions each of which has an asymptotic development, is the term by term sum of the separate asymptotic series.*

The truth of this corollary as a direct extension of the addition theorem for two series is at once obvious.

For the sake of brevity, we say that *we may add a finite number of asymptotic series.*

22. Corollary II. *The asymptotic expansion of the sum of a constant and a given asymptotically expansible function is the sum of the constant and the asymptotic expansion of the given function.*

Let $$J(x) \sim S_n + \frac{\epsilon_n}{x^n}$$

and let A be any constant. Then we have

$$\lim_{x=\infty} x^n \left| (J(x) + A) - (A + S_n) \right| = \lim_{x=\infty} \epsilon_n = 0,$$

whence it follows that

$$J(x) + A \sim A + S_n + \frac{\epsilon_n}{x^n}.$$

Q.E.D

23. Corollary III. If $J(x) \sim \sum_{n=0}^{\infty} \frac{c_n}{x^n}$, then $K \cdot J(x) \sim K \sum_{n=0}^{\infty} \frac{c_n}{x^n}$ or $\sum_{n=0}^{\infty} K \frac{c_n}{x^n}$, K being any real number.

The truth of this corollary for K any positive integer follows from corollary I by making $J_1(x) = J_2(x) = J_3(x) = \cdots = J_K(x)$. That the corollary is generally true is proven as follows:

Given $J(x) \sim c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} + \frac{\epsilon_n}{x^n}$;

To prove $K\,J(x) \sim K c_0 + \frac{K c_1}{x} + \frac{K c_2}{x^2} + \cdots + \frac{K c_n}{x^n} + \frac{K \epsilon_n}{x^n}$.

We have by hypothesis,

$$\lim_{x=\infty} x^n \left| J(x) - S_n(x) \right| = \lim_{x=\infty} \epsilon_n = 0;$$

then, as K is real,

$$\lim_{x=\infty} x^n \left| K \cdot J(x) - K \cdot S_n(x) \right| = \lim_{x=\infty} K \cdot \epsilon_n = K \lim_{x=\infty} \epsilon_n = 0;$$

whence follows the corollary.

More briefly, if both members of an asymptotic equality be multiplied by the same constant, the nature of the equality is not changed.

24. Subtraction of Asymptotic series.

Theorem. If $J(x) \sim \sum_{n=0}^{\infty} \frac{c_n}{x^n}$, and $J'(x) \sim \sum_{n=0}^{\infty} \frac{c'_n}{x^n}$, then $J(x) - J'(x) \sim \sum_{n=0}^{\infty} \frac{c_n - c'_n}{x^n}$.

Proof. Since $J'(x) \sim \sum_{n=0}^{\infty} \frac{c'_n}{x^n}$,

therefore $-J'(x) \sim \sum_{n=0}^{\infty} \frac{-c'_n}{x^n}$; by Cor. II. - Addition Th.

Then $J(x) + (-J'(x)) \sim \sum_{n=0}^{\infty} \frac{c_n + (-c'_n)}{x^n}$,

that is $J(x) - J'(x) \sim \sum_{n=0}^{\infty} \frac{c_n - c'_n}{x^n}$. Q.E.D

If the two asymptotic series $\sum_{n=0}^{\infty} \frac{c_n}{x^n}$ and $\sum_{n=0}^{\infty} \frac{c'_n}{x^n}$ used in the above theorem be equal, then $J(x) - J'(x)$ is asymp-

totically equal to the series $\sum_{n=0}^{\infty} \frac{c_n - c'_n}{x^n}$, which is identically zero. In this case it does not necessarily follow that $J(x) = J'(x)$. If $J(x) \neq J'(x)$, then $J = J(x) - J'(x)$ is a function which gives rise to an asymptotic expansion which is identically zero. Such a function has been discussed above (see §16).

25. Multiplication of Asymptotic Series.

Definition.— Let

$$\sum_{n=0}^{\infty} \frac{c_n}{x^n} = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} + \frac{\epsilon_n}{x^n} = S_n + \frac{\epsilon_n}{x^n},$$

and $$\sum_{n=0}^{\infty} \frac{c'_n}{x^n} = c'_0 + \frac{c'_1}{x} + \frac{c'_2}{x^2} + \cdots + \frac{c'_n}{x^n} + \frac{\epsilon'_n}{x^n} = S'_n + \frac{\epsilon'_n}{x^n}$$

be two asymptotic series. We then call the series

$$\sum_{n=0}^{\infty} \frac{b_n}{x^n} = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots + \frac{b_n}{x^n} + \frac{\mu_n}{x^n} = S + \frac{\mu_n}{x^n},$$

in which

$$b_0 = c_0 c'_0,$$

$$b_1 = c_0 c'_1 + c_1 c'_0,$$

$$b_2 = c_0 c'_2 + c_1 c'_1 + c_2 c'_0,$$

. . . . . .

$$b_n = c_0 c'_n + c_1 c'_{n-1} + c_2 c'_{n-2} + \cdots + c_{n-1} c'_1 + c_n c'_0,$$

$$\mu_n = \frac{c_1(c'_n + \epsilon'_n) + c_2 c'_{n-1} + \cdots + c_{n-1} c'_2 + (c_n + \epsilon_n) c'_1 + \cdots + (c_n + \epsilon_n)(c'_n + \epsilon'_n)}{x^n},$$

the product* of the two given asymptotic series.

We shall now establish the following theorem:

Theorem. The product of two asymptotic series represents asymptotically the product of the two functions

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* This product is formed according to the rules for forming the product of two convergent series. See Chrystal, Algebra, Vol. II, p. 127.

represented by these series.

Let $$J(x) \sim \sum_{n=0}^{\infty} \frac{c_n}{x^n} \equiv S_n + \frac{\epsilon_n}{x^n},$$

and $$J'(x) \sim \sum_{n=0}^{\infty} \frac{c'_n}{x^n} \equiv S'_n + \frac{\epsilon'_n}{x^n}.$$

Let $$\sum_{n=0}^{\infty} \frac{b_n}{x^n} \equiv S + \frac{\mu_n}{x^n}$$

be the product of the two asymptotic series, the product series being formed as defined above. We are then to prove that $J = J(x) \cdot J'(x)$ is represented asymptotically by the series $S + \frac{\mu_n}{x^n}$.

From the definition of S_n, S'_n, S and b_n, we have

$$S_n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n},$$

$$S'_n = c_0 + \frac{c'_1}{x} + \frac{c'_2}{x^2} + \cdots + \frac{c'_n}{x^n},$$

$$S_n S'_n = \left(c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n}\right)\left(c'_0 + \frac{c'_1}{x} + \frac{c'_2}{x^2} + \cdots + \frac{c'_n}{x^n}\right)$$

$$= b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots + \frac{b_n}{x^n} + \frac{b_{n+1}}{x^{n+1}} + \frac{b_{n+2}}{x^{n+2}} + \cdots + \frac{b_{2n}}{x^{2n}}$$

$$= S + \frac{b_{n+1}}{x^{n+1}} + \frac{b_{n+2}}{x^{n+2}} + \cdots + \frac{b_{2n}}{x^{2n}}$$

$$= S + \frac{1}{x^n}\left(\frac{b_{n+1}}{x} + \frac{b_{n+2}}{x^2} + \cdots + \frac{b_{2n}}{x^n}\right).$$

Therefore it follows that

(1) $$\lim_{x=\infty} x^n \left| S_n S'_n - S \right| = \lim_{x=\infty} \left| \left(\frac{b_{n+1}}{x} + \cdots + \frac{b_{2n}}{x^n}\right) \right| = 0.$$

We have

$$J(x) = S_n + \frac{\epsilon_n}{x^n},$$

and $$J'(x) = S'_n + \frac{\epsilon'_n}{x^n},$$

where $$\lim_{x=\infty} \epsilon_n = 0, \quad \lim_{x=\infty} \epsilon'_n = 0;$$

then $$x^n (J - S_n S'_n) = S_n \epsilon'_n + S'_n \epsilon_n + \frac{\epsilon_n \epsilon'_n}{x^n}.$$

Now $$\lim_{x=\infty} S_n \epsilon'_n = c_0 \lim_{x=\infty} \epsilon'_n = c_0 \cdot 0 = 0,$$

$$\lim_{x=\infty} S'_n \cdot \epsilon_n = c'_0 \lim_{x=\infty} \epsilon_n = c'_0 \cdot 0 = 0,$$

$$\lim_{x=\infty} \frac{\epsilon_n \epsilon'_n}{x^n} = 0, \text{ since } \lim_{x=\infty} \epsilon_n = 0, \text{ and } \lim_{x=\infty} \epsilon'_n = 0;$$

whence $$\lim_{x=\infty} \left(S_n \epsilon'_n + S'_n \epsilon_n + \frac{\epsilon_n \epsilon'_n}{x^n}\right) = 0.$$

Hence (2) $\lim_{x=\infty} x^n / J - S_n S_n' / = 0$

From (1) and (2) we have, since J, S_n, S_n' are real,

$$\lim_{x=\infty} x^n / J - S / = 0,$$

and therefore the series

$$b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots\cdots$$

asymptotically represents the product of the two functions $J(x)$ and $J(x)$. Q.E.D.

26. Corollary I. — The asymptotic representation of the product of any finite number of functions, each of which possesses an asymptotic development, is the product of the corresponding asymptotic series, the product being formed according to the definition given above.

This corollary follows at once as an extension of the above multiplication theorem.

27. Corollary II. — If $J(x) \sim S_n + \frac{\epsilon_n}{x^n}$, then $\{J(x)\}^K \sim \{S_n + \frac{\epsilon_n}{x^n}\}^K$, K being any rational real number, different from zero.

It will be necessary to consider this corollary when

(a) K is a positive integer,
(b) K is a negative integer,
(c) K is a positive rational fraction in its lowest terms,
(d) K is a negative rational fraction in its lowest terms.

(a). Let K be a positive integer. — That the corollary is true follows then at once from corollary I by making

$$J_1(x) = J_2(x) = J_3(x) = \cdots = J_K(x).$$

(b). Let K be a negative integer, say $-b$. — Here

necessary to make use of the "Division Theorem*" which will be established in a later section, and which states that "The quotient of one asymptotic series by another is permissible, provided only that the first term, that is the constant term, of the divisor series is different from zero." In the proof of this theorem, the legitimacy of the division of unity by such an asymptotic series is established

We have the identity

$$\{J(x)\}^{\kappa} \equiv \{J(x)\}^{-b} \equiv \frac{1}{\{J(x)\}^{b}},$$

b being a positive integer.

From Cor. II, (a), it follows that

$$\{J(x)\}^{b} \sim \left\{ S_n + \frac{\epsilon_n}{x^n} \right\}^{b}.$$

From the "Division Theorem", we are enabled to write

$$\frac{1}{\{J(x)\}^{b}} \sim \frac{1}{\left\{S_n + \frac{\epsilon_n}{x^n}\right\}^{b}},$$

that is,

$$\{J(x)\}^{-b} \sim \left\{S_n + \frac{\epsilon_n}{x^n}\right\}^{-b},$$

provided the first term of the asymptotic expansion of $J(x)$ is not zero, that is, provided $\lim_{x=\infty} J(x) \neq 0$. Q.E.D.

(c). Let $\kappa = \frac{p}{q}$, p and q being positive integers, prime to each other.— The only new question which arises here is that of the legitimacy of the extraction of roots of

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* See §30, p.49.

asymptotic series, that is, whether the q^{th} root of the asymptotic expansion of a given function is the asymptotic expansion of the q^{th} root of the given function. This question further resolves itself into this: does the q^{th} root of a given asymptotically expansible function have an asymptotic expansion? I shall not now attempt to answer this question completely. This would necessitate among other things a discussion of the imaginaries introduced in the extraction of roots. This much can be said with assurance: if $\{J(x)\}^{\frac{1}{q}}$ has an asymptotic expansion it is that which is obtained by extracting the q^{th} root of the asymptotic expansion of $J(x)$.

We have

$$J(x) \sim c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} + \frac{\epsilon_n}{x^n} \text{ or } S_n + \frac{\epsilon_n}{x^n}.$$

The principal q^{th} root of this series of positive integral powers of $\frac{1}{x}$ will be a series of positive integral powers of $\frac{1}{x}$, and we may write it in the form

$$c_0' + \frac{c_1'}{x} + \frac{c_2'}{x^2} + \cdots + \frac{c_n'}{x^n} + \frac{\epsilon_n'}{x^n} = S_n' + \frac{\epsilon_n'}{x^n}.$$

We say then that if $\{J(x)\}^{\frac{1}{q}}$ has an asymptotic expansion, we have

$$\{J(x)\}^{\frac{1}{q}} \sim S_n' + \frac{\epsilon_n'}{x^n}.$$

For since

$$J(x) \equiv \left[\{J(x)\}^{\frac{1}{q}}\right]^q,$$

and since

$$S_n + \frac{\epsilon_n}{x^n} \equiv \left(S_n' + \frac{\epsilon_n'}{x^n}\right)^q,$$

then by virtue of Cor. II, (a), of this theorem

$$J(x) \sim \left(S_n' + \frac{\epsilon_n'}{x^n}\right)^q,$$

that is,

$$J(x) \sim S_n + \frac{\epsilon_n}{x^n},$$

which is in agreement with our original hypothesis, namely, that

$$J(x) \sim S_n + \frac{\epsilon_n}{x^n}.$$

Then, since

$$\{J(x)\}^{\frac{p}{q}} \equiv \left[\{J(x)\}^{\frac{1}{q}}\right]^p$$

and since

$$\{J(x)\}^{\frac{1}{q}} \sim \left\{S_n + \frac{\epsilon_n}{x^n}\right\}^{\frac{1}{q}},$$

we have

$$\{J(x)\}^{\frac{p}{q}} \sim \left\{S_n + \frac{\epsilon_n}{x^n}\right\}^{\frac{p}{q}},$$

by virtue of Cor. II, (a), of this theorem. Q.E.D.

(d) *Let* $K = -\frac{p}{q}$, *p and q being as in* (c). — This corollary follows at once by virtue of (b) and (c) with the assumptions therein made, namely, that $\lim_{x=\infty} J(x) \neq 0$, and that $\{J(x)\}^{\frac{1}{q}}$ has an asymptotic expansion.

The above theorem and corollaries enable us to say briefly that *we may multiply together any finite number of asymptotic series, and that we may raise an asymptotic series to any rational power.*

28. *A Rational Integral Function of an Asymptotically Expansible Function.*

Theorem. *If* $J(x)$ *has an asymptotic expansion, then any rational integral function of* $J(x)$ *has for its*

asymptotic expansion the same rational integral function of the asymptotic expansion of $J(x)$

given $f(w) = A_0 + A_1 w + A_2 w^2 + \cdots + A_p w^p$, ($p$ finite),

and $J \sim c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n} + \frac{\epsilon_n}{x^n}$,

or $J \sim S_n + \frac{\epsilon_n}{x^n}$;

To prove that

$$f(J) \sim A_0 + A_1\left(S_n + \frac{\epsilon_n}{x^n}\right) + A_2\left(S_n + \frac{\epsilon_n}{x^n}\right)^2 + \cdots + A_p\left(S_n + \frac{\epsilon_n}{x^n}\right)^p.$$

Proof. We have

$$f(J) = A_0 + A_1 J + A_2 J^2 + \cdots + A_p J^p.$$

From the "Multiplication Theorem", Cor. II, we have

$$A_m J^m \sim A_m \left(S_n + \frac{\epsilon_n}{x^n}\right)^m \quad (m = 1, 2, 3, \cdots, p);$$

From the "Addition Theorem", Cors. II and III, we have

$$A_0 + \sum_{m=1}^{p} A_m J^m \sim A_0 + \sum_{m=1}^{p} A_m \left(S_n + \frac{\epsilon_n}{x^n}\right)^m;$$

therefore

$$f(J) \sim A_0 + A_1\left(S_n + \frac{\epsilon_n}{x^n}\right) + A_2\left(S_n + \frac{\epsilon_n}{x^n}\right)^2 + \cdots + A_p\left(S_n + \frac{\epsilon_n}{x^n}\right)^p.$$

Q.E.D.

29. A Power Series of a Function having an Asymptotic Expansion. - We may go further than indicated in the last article and announce the following theorem.

Theorem. Let $J(x) \sim S_n + \frac{\epsilon_n}{x^n}$,

where $S_n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n}$,

and let the series for $f(w)$,

$$A_0 + A_1 w + A_2 w^2 + A_3 w^3 + \cdots + A_p w^p + \cdots$$

be convergent for

$|w| = \rho$, where $\rho > |c_0|$,

then $f(J) = A_0 + A_1 J + A_2 J^2 + A_3 J^3 + \cdots + A_p J^p + \cdots$

will be some function of x, say $F(x)$, such that we shall have

$$F(x) = A_0 + A_1\left(S_n + \frac{\epsilon_n}{x^n}\right) + A_2\left(S_n + \frac{\epsilon_n}{x^n}\right)^2 + \cdots + A_p\left(S_n + \frac{\epsilon_n}{x^n}\right)^p + \cdots,$$

wherein the indicated expansions are to be performed in accordance with the rules and definitions previously given.

Since $\lim_{x=\infty} J(x) = c_0$, and since the series for $f(w)$ is convergent for $|w| = \rho$, $\rho > |c_0|$, therefore for sufficiently large values of x the series for $f(J)$ will be convergent, that is, the series for $F(x)$

$$A_0' + \frac{A_1'}{x} + \frac{A_2'}{x^2} + \cdots + \frac{A_n'}{x^n} + \cdots$$

will be convergent for sufficiently large values of x. We are then to show that the asymptotic expansion of this function $F(x)$ may be obtained by substituting for $J(x)$ its asymptotic expansion in the series for $f(J)$.

The series for $f(w)$,

$$(1) \qquad A_0 + A_1 w + A_2 w^2 + \cdots + A_p w^p + \cdots = \lim_{n=\infty} S_n(w)$$

converges for $|w| = \rho$. It therefore converges uniformly* within the interval for which

$$|w| < \rho > |c_0|.$$

Now $$F(x) = \lim_{n=\infty} S_n(J(x));$$

whence it follows that

$$(2) \qquad \lim_{x=\infty} F(x) = \lim_{x=\infty} \lim_{n=\infty} S_n(J(x)).$$

Because (1) converges uniformly for the interval for which

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* See Hawkness and Morley, Introduction to Analytic Functions, 1898, p. 129.

$$|J(x)| < \rho, \quad \rho > |C_0|,$$

we may interchange the order of taking limits in (2) since $\lim_{x=\infty} J(x) = C_0$, and write

$$\begin{aligned}\lim_{x=\infty} F(x) &= \lim_{x=\infty} \lim_{n=\infty} S_n(J(x)) \\ &= \lim_{n=\infty} \lim_{x=\infty} S_n(J(x)) \\ &= A_0 + A_1 C_0 + A_2 C_0^2 + A_3 C_0^3 + \cdots \\ &= A_0 + \sum_{p=1}^{\infty} A_p C_0^p \\ &= C_0, \text{ say.}\end{aligned}$$

Then we have

$$F(x) - C_0 = A_1(J(x) - C_0) + A_2(J(x)^2 - C_0^2) + \cdots + A_p(J(x)^p - C_0^p) + \cdots.$$

Now let us calculate

$$\lim_{x=\infty} x\,|F(x) - C_0|.$$

We have to calculate for this purpose expressions of the type

$$\lim_{x=\infty} x\,|J(x)^p - C_0^p|, \qquad (p = 1, 2, 3, \cdots, p, \cdots).$$

We have

$$\begin{aligned}J(x)^p &= \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots\right)^p \\ &= C_0^p + \frac{p\,C_0^{p-1} C_1}{x} + \cdots;\end{aligned}$$

whence

$$\lim_{x=\infty} x\,|J(x)^p - C_0^p| = p\,C_0^{p-1} C_1, \qquad (p = 1, 2, 3, \cdots, p).$$

We have therefore

$$\begin{aligned}\lim_{x=\infty} x\,|F(x) - C_0| &= \sum_{p=1}^{\infty} A_p\, x\,|J(x)^p - C_0^p| \\ &= \sum_{p=1}^{\infty} A_p\, p \cdot C_0^{p-1} C_1 \\ &= C_1,\end{aligned}$$

where $C_1 = C_1(A_1 + 2A_2 C_0 + 3A_3 C_0^2 + \cdots + p\,A_p C_0^p + \cdots)$,

a series which is evidently convergent because of the hypothesis made concerning the interval of convergence of the series for $f(J)$. From the equation involving C_1, we have

$$\lim_{x=\infty} x\left| F(x) - \left(C_0 + \frac{C_1}{x}\right)\right| = 0.$$

Proceeding in this manner we obtain

$$\lim_{x=\infty} x^n\left| F(x) - \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots + \frac{C_n}{x^n}\right)\right| = 0,$$

for all values of n, and consequently the function $F(x)$ has an asymptotic development which is obtained by replacing $J(x)$ by its asymptotic expansion in the series for $f(J)$, and effecting the calculations by the methods previously defined.

Q.E.D.

30. Division of Asymptotic Series.

Theorem. — The asymptotic expansion of the quotient of two functions, each of which has an asymptotic expansion, is the quotient of the two asymptotic expansions, provided only that the constant term in the asymptotic expansion of the divisor is not zero.

Let $$J(x) \backsim C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots + \frac{C_n}{x^n} + \frac{\epsilon_n}{x^n} \qquad (C_0 \neq 0)$$

$$= C_0(1+\phi),$$

where $$\phi = \frac{C_1}{C_0 x} + \frac{C_2}{C_0 x^2} + \frac{C_3}{C_0 x^3} + \cdots + \frac{C_n + \epsilon_n}{C_0 x^n}.$$

Then we have

$$\lim_{x=\infty} \phi = \frac{1}{C_0} \lim_{x=\infty} \left(J(x) - C_0\right) = 0.$$

Now let us consider the function $\frac{1}{J(x)}$. We have

$$\frac{1}{J(x)} = \frac{1}{C_0(1+\phi)} = \frac{1}{C_0}\left(1 - \phi + \phi^2 - \phi^3 + \cdots\right),$$

for all values of Φ which make the series convergent. The series

$$1 - \Phi + \Phi^2 - \Phi^3 + \ldots .$$

is convergent ~~for~~ $|\Phi| < 1$. But we have shown above that, for sufficiently large values of x, Φ approaches the limit zero. Consequently for these sufficiently large values of x, the series for $\frac{1}{J(x)}$ is absolutely convergent, since it converges for these values of x when all the terms are made positive. Since the radius of convergence of the Φ series $\rho_\Phi = 1$, and since the $\lim_{x=\infty} \Phi = 0 = c_0'$, that is, since the first term in the asymptotic expansion of the function Φ is $c_0' = 0$, and since therefore $\rho_\Phi > |c_0'|$, consequently by virtue of the immediately preceding theorem (§29), we may obtain the asymptotic expansion of $\frac{1}{J(x)}$ by substituting for Φ its asymptotic expansion in the Φ series corresponding to $\frac{1}{J(x)}$ and effecting the calculations by the methods previously given.

We have thus shown that the asymptotic expansion of the quotient obtained by dividing unity by $J(x)$ is the quotient obtained by dividing unity by the asymptotic expansion of $J(x)$.

Since $\frac{J'(x)}{J(x)} \equiv J'(x) \cdot \frac{1}{J(x)}$ and since, if we let $J(x) \sim S_n + \frac{\epsilon_n}{x^n}$, we have $\frac{1}{J(x)} \sim \frac{1}{S_n + \frac{\epsilon_n}{x^n}}$ on the condition only that $c_0 \neq 0$, therefore if we let $J'(x) \sim S_n' + \frac{\epsilon_n'}{x^n}$, we have by virtue of the "Multiplication Theorem",

$$\frac{J'(x)}{J(x)} \sim \frac{S_n' + \frac{\epsilon_n'}{x^n}}{S_n + \frac{\epsilon_n}{x^n}}. \quad \text{Q.E.D.}$$

We may express this briefly by saying that we may divide one by the other of two asymptotic series provided the first term c_0 of the divisor series is not zero.

31. In an article* on "The asymptotic expansion of an analytic function" Mr. E. Cunningham, St. John's College, Cambridge, England, has considered the asymptotic expansion of the quotient of two asymptotically expansible functions, the expansions being valid in the neighborhood of the origin. Since his method of attack is somewhat different from that used in the preceding article and since the expansions are valid for very small values of x (not necessarily restricted to being real), we shall consider his treatment, revised and corrected in some parts.

According to his definition, the function $\phi(x)$ is said to be asymptotically represented by the series

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

within a given angle S at the point $x = 0$, if a number ρ exists, such that at all points within both S and the circle $|x| = \rho$, ρ being sufficiently small,

$$\left| \frac{\phi(x) - (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)}{x^n} \right| < \epsilon,$$

ϵ being any assignable small number.

We propose now to prove that if $\phi(x)$ is represented asymptotically by the series

* See Mess. of mathematics, vol. 34, 1905, pp. 155-171

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots, \qquad (a_0 \neq 0),$$

within the region S as above defined, then $\frac{1}{\varphi(x)}$ will be represented asymptotically by the formal expansion of the quotient of unity by this series, provided $a_0 \neq 0$.

It will be sufficient to prove that if, by a formal expansion of the expression

$$\frac{1}{a_0 + a_1 x + \cdots + a_n x^n + \cdots},$$

the series

$$b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$$

be obtained, then within the region S the function $\frac{1}{\varphi(x)}$ is represented asymptotically by the b- series.

We have the asymptotic equality

$$\varphi(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \epsilon_n x^n, \qquad (a_0 \neq 0),$$

where $\epsilon_n = a_{n+1} x + a_{n+2} x^2 + \cdots,$

valid within the region σ, common to the angle S and the circle $|x| = \rho$, ρ being sufficiently small so that $|\epsilon_n| < \epsilon$, ϵ being an arbitrarily small positive number.

Thus within σ,

$$\frac{1}{\varphi(x)} \equiv \frac{1}{a_0 + a_1 x + \cdots + (a_n + \epsilon_n) x^n}$$

$$\equiv \frac{1}{a_0 + a_1 x + \cdots + a_n x^n} - \frac{\epsilon_n}{(a_0 + a_1 x + \cdots + a_n x^n)(a_0 + a_1 x + \cdots + (a_n + \epsilon_n) x^n)} x^n.$$

Now since $a_0 \neq 0$, the roots of the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

all lie exterior to the circle about the origin, $|x| = \rho'$, (ρ' finite).

We have therefore

(1) $$|a_0 + a_1 x + \cdots + a_n x^n| > c$$

valid for all points within the circle $|x| = \rho'$, and therefore valid within the region common to this circle and the angle S, that is, within the region common to σ and the circle $|x| = \rho'$. If $\rho' \geqq \rho$, this last-named region is σ; if $\rho' < \rho$, this region is $\sigma' < \sigma$, and consequently σ contains σ'.

Further, since within the region σ,

(2) $$|\epsilon_n| < \epsilon, \qquad (\epsilon \text{ arbitrarily small}),$$

and since $a_0 \neq 0$, we see that, for all points of some region common to σ' and some finite circle $|x| = \rho''$ about the point $x = 0$, which region is either identical with σ', or is σ'' where σ' contains σ'',

(3) $$|a_0 + a_1 x + \cdots + (a_n + \epsilon_n) x^n| > 0,$$

but has a finite lower limit.

Now let ρ_1 be equal to the least of the radii ρ, ρ', ρ'', and let σ_1 be the region common to the angle S and the circle $|x| = \rho_1$, drawn about the point $x = 0$. Then, since inequalities (1), (2), (3) are valid for all points within their respective regions $\sigma', \sigma, \sigma''$, they will be valid within this region σ_1, determined by the angle S and the circle of radius ρ_1 about the origin as centre.

We have therefore

(1) $|a_0 + a_1 x + \cdots + a_n x^n| > 0$, but finite,*

(2) $|\epsilon_n| < \epsilon$, (ϵ arbitrarily small and positive),

(3) $|a_0 + a_1 x + \cdots + (a_n + \epsilon_n) x^n| > 0$, but finite,*

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* (1) and (3) have a finite lower limit.

all valid for all points within the region σ_1. We have then

(4) $|a_0+a_1x+\cdots+a_nx^n|\cdot|a_0+a_1x+\cdots+(a_n+\epsilon_n)x^n| > A'$,

where A' is finite and $\delta > 0$. From (2) and (4) it follows that

$$\left|\frac{\epsilon_n}{(a_0+a_1x+\cdots+a_nx^n)\{a_0+a_1x+\cdots+(a_n+\epsilon_n)x^n\}}\right| < \frac{\epsilon}{A'},$$

or (5) $$\left|\frac{\epsilon_n}{(a_0+a_1x+\cdots+a_nx^n)\{a_0+a_1x+\cdots+(a_n+\epsilon_n)x^n\}}\right| < A\epsilon,$$

where A is finite and $\delta > 0$, and ϵ is an arbitrarily small positive number. Inequality (5) is valid for the region σ_1.

Since within σ_1 the polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

does not vanish, the function

$$\frac{1}{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}$$

may be expanded in a convergent series of integral powers of x, which series may be written in the form

$$b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + x^{n+1}(c_{n+1} + c_{n+2}x + \cdots),$$

that is the expansion of the above function will be identical with the b-series mentioned above (see p. 52), certainly as far as the term involving x^n; and the function

$$c_{n+1} + c_{n+2}x + c_{n+3}x^2 + \cdots$$

will converge and be less than C in absolute value, C {finite, positive}, for the region σ_1.

Thus

$$\left|\frac{1}{a_0+\cdots+a_nx^n} - (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)\right| < |x^{n+1}|\,C,$$

and for $|x| < \eta$, this becomes

(6) $$\left|\frac{1}{a_0+a_1x+\cdots+a_nx^n} - (b_0+b_1x+b_2x^2+\cdots+b_nx^n)\right| < C\eta|x|^n.$$

Since from the identity involving $\frac{1}{\phi(x)}$ (See p. 52), we have

$$(7)\qquad \left|\frac{1}{\phi(x)} - \frac{1}{a_0+\cdots+a_n x^n}\right| = \left|\frac{\epsilon_n}{(a_0+\cdots+a_n x^n)\{a_0+\cdots+(a_n+\epsilon_n)x^n}\cdot x^n\right|,$$

it therefore follows from (5), (6) and (7) that

$$\left|\frac{1}{\phi(x)} - (b_0+b_1x+\cdots+b_n x^n)\right| < (A\epsilon + C\eta)\cdot|x|^n.$$

Finally then we have

$$\left|\frac{\frac{1}{\phi(x)} - (b_0+b_1x+\cdots+b_n x^n)}{x^n}\right| < A\epsilon + C\eta.$$

Now ϵ and η are arbitrarily small positive numbers, and A and C are finite positive constants, whence we have

$$\lim_{\{x=0 \text{ within } \sigma_1\}} \left|\frac{\frac{1}{\phi(x)} - (b_0+b_1x+\cdots+b_n x^n)}{x^n}\right| = 0,$$

that is, within the region τ_1, we have that the series

$$b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$$

represents the function $\frac{1}{\phi(x)}$ asymptotically.

Now regions σ, σ', σ'' and σ_1 had the angle S in common, and since the conclusions throughout have been shown to be valid for all points within that region σ_1 determined by the angle S and the least of the radii ρ, ρ', ρ'', namely ρ_1, and since ρ was to be considered sufficiently small in order to have the asymptotic expansion for $\phi(x)$ valid, therefore let ρ be equal to ρ_1, the least of the three radii ρ, ρ', ρ''. The region of validity of the original asymptotic expansion is then the region of validity of each of the inequalities (1), (2) and (3)

above, and consequently also the region of validity of the series which asymptotically represents the function $\frac{1}{f(x)}$. The theorem is therefor proved.

The reasoning in the above proof does not depend on the fact that the variable x is complex and requires but little change in symbolism and wording to become a proof for the real variable.

32. A Double Powers Series of an Asymptotically Expansible Function.— We are now able to extend the theorem of §27, and to state the following theorem for a double power series.

Theorem. Let $$J(x) \sim S_n + \frac{\epsilon_n}{x^n},$$ where $$S_n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_n}{x^n},$$ and let the series for $f(w)$, $$\cdots + A_{-2}w^{-2} + A_{-1}w^{-1} + A_0 + A_1 w + A_2 w^2 + \cdots$$ be convergent for $$\rho_1 \leqq |w| \leqq \rho_2,$$ where $$\rho_1 < \{|\tfrac{1}{c_0}| \text{ and } |c_0|\}\rho_2;$$ then $f(J)$ is some function of x, say $F(x)$, such that $F(x)$ is represented asymptotically by the series formed by replacing $J(x)$ in $f(J)$ by its asymptotic expansion and effecting the calculations by the previously defined methods.

Let $$f(w) = f_1(w) + f_2(w),$$ where $$f_1(w) = \sum_{p=-\infty}^{-1} A_p w^p,$$ and $$f_2(w) = \sum_{p=0}^{\infty} A_p w^{-p}.$$

Let $f(J)$, $f_1(J)$, $f_2(J)$ as functions of x be designated by $F(x)$, $F_1(x)$ and $F_2(x)$, respectively.

The series for $f_1(w)$ and $f_2(w)$ will be convergent for $|w| \geq \rho_1$, and $|w| \leq \rho_2$, respectively. Since $\lim_{x=\infty} J(x) = c_0$, and since $\rho_1 < |c_0| < \rho_2$, by hypothesis, therefore for sufficiently large values of x, the series for $f(J)$, $f_1(J)$, $f_2(J)$ and consequently the series for $F(x)$, $F_1(x)$, $F_2(x)$ will each be convergent. We propose to show that the asymptotic expansion of $F(x)$ is obtained as indicated in the theorem by treating the asymptotic expansions of $F_1(x)$, $F_2(x)$ and $F(x) \equiv F_1(x) + F_2(x)$, respectively.

In the theorem of article 23, we have shown that

$$F_2(x) \sim A_0 + A_1\left(S_n + \frac{\epsilon_n}{x^n}\right) + A_2\left(S_n + \frac{\epsilon_n}{x^n}\right)^2 + \cdots,$$

wherein the indicated expansions are to be performed according to the previously given methods.

Since the asymptotic expansion of $F(x) \equiv F_1(x) + F_2(x)$ can, by virtue of the "Addition Theorem," be obtained from the asymptotic expansions of $F_1(x)$ and $F_2(x)$, it remains only to obtain the asymptotic expansion of $F_1(x)$.

Now, since $\rho_1 < |c_0| < \rho_2$, therefore $c_0 \neq 0$ and, the condition of the "Division Theorem" being fulfilled, we obtain the asymptotic expansion of J^{-1} as the quotient of unity by $c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots$, the first term of the resulting asymptotic expansion being $\frac{1}{c_0}$.

Then, since $\lim_{x=\infty} (J(x)^{-1}) = \frac{1}{c_0}$, and since $\rho_1 < \left|\frac{1}{c_0}\right| < \rho_2$,

$$f_1(J) = \sum_{p=-1}^{-\infty} A_p J^p = \sum_{p=1}^{\infty} A_{-p} (J^{-1})^p$$

is a function of x, $F_1(x)$, whose asymptotic expansion is obtained, by virtue of the theorem of article 28, by substituting for $J^{-1}(x)$ in $f_1(J)$ its asymptotic expansion and effecting the simplification calculations by the previously defined methods. Since $J^{-1}(x) \equiv \frac{1}{J(x)}$, and since the asymptotic expansion of $J^{-1}(x)$ was obtained by replacing $J(x)$ in $\frac{1}{J(x)}$ by its asymptotic expansion, therefore we have obtained the asymptotic expansion of $F_1(x)$ by substituting for $J(x)$ in $f_1(J)$ its asymptotic expansion and effecting the calculations as indicated above.

We have now

$$F_1(x) \sim \sum_{p=-\infty}^{-1} A_p \left(S_n + \frac{\epsilon_n}{x^n}\right)^p,$$

$$F_2(x) \sim \sum_{p=0}^{\infty} A_p \left(S_n + \frac{\epsilon_n}{x^n}\right)^p,$$

$$F(x) \equiv F_1(x) + F_2(x);$$

then by the "Addition Theorem", we have $F(x)$ asymptotically represented by the series formed by effecting the indicated operations upon the series

$$\sum_{p=-\infty}^{-1} A_p \left(S_n + \frac{\epsilon_n}{x^n}\right)^p + \sum_{p=0}^{\infty} A_p \left(S_n + \frac{\epsilon_n}{x^n}\right)$$

according to the methods previously explained for expanding asymptotic series. Q.E.D.

The introduction into this theorem of the double condition requiring that both $|c_0|$ and $\left|\frac{1}{c_0}\right|$ lie between

ρ_1 and ρ_2 somewhat simplified the proof. The more general theorem in which it is required that $\rho_1 < |c_0| < \rho_2$, no further restrictions being placed upon $|\frac{1}{c_0}|$, is true. The proof which I have for it is somewhat longer and more involved.

The two theorems indicated are true for the complex variable, the proofs for which follow very closely the proofs for the real variable.

33. Any Rational Function of Functions having Asymptotic Expansions. — From the preceding articles we are enabled to state the following theorem:

Theorem. If any finite number of functions $f, \Phi, \theta, \ldots$ have asymptotic expansions valid for the same values of the variable, then (i) any rational integral function, (ii) any rational function whatever, of one or more of the functions $f, \Phi, \theta, \ldots$ is represented asymptotically for the same values of the variable by the development in series of the same function of the several corresponding asymptotic expansions, provided no factor of the denominator in case (ii) vanishes for the values of the variable for which the asymptotic expansions of $f, \Phi, \theta, \ldots$ hold.

Proof. (i). The general term of a rational integral function of f, Φ, θ will be of the form $A f^m \Phi^n \theta^p$, where A is constant, and m, n, p are positive integers or zero. The asymptotic expansion of f^m is the m^{th} power of

the asymptotic expansion of f; similarly the asymptotic expansions of Φ^n and θ^p are obtained. Then the asymptotic expansion of $A \cdot f^m \cdot \Phi^n \cdot \theta^p$ is the product of A and the asymptotic expansions of f^m, Φ^n and θ^p. Since the rational integral function is the sum of a finite number of such terms as that just considered, its asymptotic expansion is the sum of the asymptotic expansions of the several terms. This establishes (i) for the three functions f, Φ, and θ. The extension to any finite number of functions is obviously valid.

(ii). Let $R(f, \Phi, \theta, \cdots)$ be any rational function whatever of f, Φ, θ, $\cdots$. Then we may write

$$R(f, \Phi, \theta, \cdots) = \frac{F(f, \Phi, \theta, \cdots)}{H(f, \Phi, \theta, \cdots)},$$

where $F(f, \Phi, \theta, \cdots)$ and $H(f, \Phi, \theta, \cdots)$ are each rational integral functions of f, Φ, θ, $\cdots$. Their asymptotic expansions are, by (i), obtained as the developments of the same functions of the asymptotic expansions of f, Φ, θ, $\cdots$. By the "Division Theorem", provided the first term in the asymptotic expansion of H is different from zero, $R(f, \Phi, \theta, \cdots)$ is represented asymptotically by the development in a series of the quotient of the asymptotic series corresponding to F by that corresponding to H. This establishes (ii).

34. Resumé of Functions having Asymptotic Expansions. We have now treated in detail several methods by which may be formed functions which admit asymptotic expansions and the methods of obtaining these expansions.

Having the asymptotic expansion of a given function $J(x)$, we have shown that the several functions

(a) $K \cdot J(x)$ (K real and finite),

(b) $\{J(x)\}^K$ (K rational, $K \neq 0$),

(c) $A_0 + J(x)$ (A_0 any constant),

(d) $\sum_{p=0}^{P} A_p J^p$ (A_p real and finite, p integral),

(e) $\sum_{p=0}^{\infty} A_p J^p$ $\left\{ \begin{array}{l} A_p \text{ real and finite, } p \text{ integral, the series} \\ \sum_{p=0}^{\infty} A_p w^p \text{ convergent for } |w| = \rho, \rho > |c_0|, \\ c_0 \text{ being the first term of the asymptotic expansion of } J(x). \end{array} \right\}$,

(f) $\sum_{p=-\infty}^{\infty} A_p J^p$ $\left\{ \begin{array}{l} A_p \text{ real and finite, } p \text{ integral, the series} \\ \sum_{p=-\infty}^{\infty} A_p w^p \text{ convergent for } \rho_1 \leqq |w| \leqq \rho_2, \\ \rho_1 < \left\{ \left| \frac{1}{c_0} \right| \text{ and } |c_0| \right\} < \rho_2, \ c_0 \text{ as in (e) above} \end{array} \right\}$,

have each, with perhaps certain modified but expressed conditions, an unique asymptotic expansion which is obtained by replacing $J(x)$ by its asymptotic expansion and effecting the necessary simplification calculations according to rules which are in general the same as those which apply to similar calculations involving convergent series.

Again we have shown that if we have the asymp-

totic expansions of $J_1(x)$, $J_2(x)$, ..., $J_m(x)$, (m finite), valid for the same values of the variable, then the associative and distributive laws hold true for their sums, their differences, their products, and their quotients, the last with the restrictive condition that the first term of the asymptotic expansion of the divisor series shall not be zero.

E.W. Barnes has made (e) of the above set the basis of attack for securing the asymptotic expansion of a large class of transcendental integral functions. He has in most instances considered the problem for the complex variable and has been able in most cases to define the sector for which the obtained asymptotic expansion is valid.

35. Integration of Asymptotic Expansions. We may now state the following theorem concerning the term-wise integration of an expansion which represents a function asymptotically.

Theorem. Let $J(x)$ be represented asymptotically by the series

$$c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots + \frac{c_m}{x^m} + \cdots,$$

then we shall have the asymptotic equality

$$\int_x^\infty \left[J(x) - c_0 - \frac{c_1}{x} \right] dx = \frac{c_2}{x} + \frac{c_3}{2x^2} + \frac{c_4}{3x^3} + \cdots + \frac{c_m}{(m-1)x^{m-1}} + \cdots.$$

See Barnes E. W., "A Memoir on Integral Functions," Phil. Trans. of the Royal Society of London, Series A, vol 199, 1902, pp. 411-500.

We have here transposed the first two terms of the series for $J(z)$ before integrating in order not to introduce meaningless integrals.

We may write

$$J(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_m + \epsilon_m}{z^m},$$

where

$$\epsilon_m = \frac{c_{m+1}}{z} + \frac{c_{m+2}}{z^2} + \frac{c_{m+3}}{z^3} + \cdots.$$

Integrating we have

$$\int_x^X \left[J(z) - c_0 - \frac{c_1}{z}\right] dz = \int_x^X \left(\frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots + \frac{c_m}{z^m}\right) dz + \int_x^X \frac{\epsilon_m}{z^m} dz.$$

Now since ϵ_m, and therefore also $\int_x^X \frac{\epsilon_m}{z^m} dz$ approach the limit zero when z is indefinitely increased, the second member of the last equation tends toward a limit when X increases indefinitely. This is then true of the first member and we may write

$$\int_x^\infty \left[J(z) - c_0 - \frac{c_1}{z}\right] dz = \left\{-\frac{c_2}{z} - \frac{1}{2}\frac{c_3}{z^2} - \frac{1}{3}\frac{c_4}{z^3} - \cdots - \frac{1}{m-1}\frac{c_m}{z^{m-1}} - \frac{\eta_m}{z^{m-1}}\right\}\Bigg|_x^\infty$$

$$= \frac{c_2}{x} + \frac{1}{2}\frac{c_3}{x^2} + \frac{1}{3}\frac{c_4}{x^3} + \cdots + \frac{1}{m-1}\frac{c_m}{x^{m-1}} + \frac{\eta_m}{x^{m-1}}$$

where $\lim_{x=\infty} \eta_m = 0$.

In fact, if μ_m is the greatest of the values assumed by ϵ_m for values of z between x and $+\infty$, we have

$$\left|\int_x^\infty \frac{\epsilon_m}{z^m} dz\right| < \mu_m \int_x^\infty \frac{dz}{z^m}.$$

Then

$$\left|\int_x^\infty \frac{\epsilon_m}{z^m} dz\right| = \frac{\eta_m}{x^{m-1}} < \frac{\mu_m}{m-1}\frac{1}{x^{m-1}},$$

from which inequality we have

$$\eta_m = \theta \frac{\mu_m}{m-1} \qquad |\theta| < 1,$$

whence $\lim_{x=\infty} \eta_m = 0.$

Consequently, the integral

$$\int_x^\infty \left[J(z) - c_0 - \frac{c_1}{z} \right] dz$$

has for its asymptotic expansion the series indicated.

Now we may write the following identity

$$\int_x^\infty \left[J(z) - c_0 - \frac{c_1}{z} \right] dz \equiv \int_{x_0}^\infty \left[J(z) - c_0 - \frac{c_1}{z} \right] dz - \int_{x_0}^x \left[J(z) - c_0 - \frac{c_1}{z} \right] dz.$$

We have also the following integral simplifications

$$\int_{x_0}^\infty \left[J(z) - c_0 - \frac{c_1}{z} \right] dz = \int_{x_0}^\infty \left(\frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \right) dz$$

$$= \left\{ - \frac{c_2}{z} - \frac{1}{2} \frac{c_3}{z^2} - \frac{1}{3} \frac{c_4}{z^3} - \cdots \right\} \Big|_{x_0}^\infty$$

$$= \frac{c_2}{x_0} + \frac{1}{2} \frac{c_3}{x_0^2} + \frac{1}{3} \frac{c_4}{x_0^3} + \cdots$$

$$= C_0, \text{ a constant.}$$

$$\int_{x_0}^x \left[J(z) - c_0 - \frac{c_1}{z} \right] dz = \int_{x_0}^x J(z) dz - \int_{x_0}^x c_0 dz - \int_{x_0}^x \frac{c_1}{z} dz$$

$$= \int_{x_0}^x J(z) dz - c_0 x + c_0 x_0 - c_1 \log x + c_1 \log x_0$$

$$= \int_{x_0}^x J(z) dz - c_0 x - c_1 \log x + c_0 x_0 + c_1 \log x_0.$$

$$\int_{x_0}^x J(z) dz = \int_{x_0}^x \left(c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \right) dz$$

$$= \left\{ c_0 z + c_1 \log z - \frac{c_2}{z} - \frac{1}{2} \frac{c_3}{z^2} - \frac{1}{3} \frac{c_4}{z^3} - \cdots \right\} \Big|_{x_0}^x$$

$$= c_0 x + c_1 \log x - \frac{c_2}{x} - \frac{1}{2} \frac{c_3}{x^2} - \frac{1}{3} \frac{c_4}{x^3} - \cdots$$

$$- c_0 x_0 - c_1 \log x_0 + \frac{c_2}{x_0} + \frac{1}{2} \frac{c_3}{x_0^2} + \frac{1}{3} \frac{c_4}{x_0^3} + \cdots$$

$$= c_0 x + c_1 \log x + C - \frac{c_2}{x} - \frac{1}{2} \frac{c_3}{x^2} - \frac{1}{3} \frac{c_4}{x^3} - \cdots.$$

From the above identity and the expressions for the several integrals just written, we have asymptotically

$$\int_x^\infty \left[J_z - c_0 - \frac{c_1}{z}\right] dz = \frac{c_2}{x_0} + \frac{1}{2}\frac{c_3}{x_0^2} + \frac{1}{3}\frac{c_4}{x_0^3} + \cdots$$
$$- c_0 x - c_1 \log x + \frac{c_2}{x} + \frac{1}{2}\frac{c_3}{x^2} + \frac{1}{3}\frac{c_4}{x^3} + \cdots$$
$$+ c_0 x_0 + c_1 \log x_0 - \frac{c_2}{x_0} - \frac{1}{2}\frac{c_3}{x_0^2} - \frac{1}{3}\frac{c_4}{x_0^3} - \cdots$$
$$+ c_0 x + c_1 \log x - c_0 x_0 - c_1 \log x_0$$
$$= \frac{c_2}{x} + \frac{1}{2}\frac{c_3}{x^2} + \frac{1}{3}\frac{c_4}{x^3} + \cdots,$$

as required by the theorem. We may therefore integrate an asymptotic series term by term.

36. A Generalization of the Definition of an Asymptotic Expansion. - The expression obtained in the last article for the integral $\int_{x_0}^{x} J_{(z)}\, dz$,

$$\int_{x_0}^{x} J_{(z)}\, dz = c_0 x + c_1 \log x + C - \frac{c_2}{x} - \frac{1}{2}\frac{c_3}{x^2} - \frac{1}{3}\frac{c_4}{x^3} - \cdots,$$

wherein the value of C is to be obtained from the series

$$C = -c_0 x_0 - c_1 \log x_0 + \frac{c_2}{x_0} + \frac{1}{2}\frac{c_3}{x_0^2} + \frac{1}{3}\frac{c_4}{x_0^3} + \cdots$$

by assigning a particular value to x_0, represents $\int_{x_0}^{x} J_{(z)}\, dz$ asymptotically. It is to be noted that this asymptotic expansion does not begin with a constant term, as has been customary, but with the two terms

$$c_0 x + c_1 \log x$$

which increase indefinitely with x.

This involves a generalization of the idea of asymptotic development which we are quite naturally led to introduce in the theory. For example,

it will be recalled that in obtaining Stirling's series (see §15), we had

$$\log \Gamma(z+1) = (z+\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \omega(z),$$

and that we found an asymptotic expansion for $\omega(z)$, which we substituted for $\omega(z)$ and wrote asymptotically

$$\log \Gamma(z+1) = (z+\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{2n(2n-1)} \frac{1}{z^{2n-1}}.$$

We are thus led to make the following statement:

If $\varphi(x)$, $F(x)$ and $G(x)$ are three functions of x, then, in a general manner, to say that we have the asymptotic equality

$$\varphi(x) = F(x) + G(x)\left(c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right)$$

indicates that we have the asymptotic equality

$$\frac{\varphi(x) - F(x)}{G(x)} = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots.$$

37. Differentiation of Asymptotic Series. - Although we have just shown that the integral of an asymptotically expansible function possesses an asymptotic expansion, obtained as the term-wise integral of the asymptotic expansion of the given function, yet we shall now show that the operation of differentiating an asymptotic series is not always a permissible one. To do this it is sufficient to consider a function which has an asymptotic expansion whereas its derivative has no such expansion.

For example, let us consider the function

$$F(x) = e^{-x}\sin(e^{x^2}).$$

Here we have

$$\lim_{x=\infty} F(x) = 0 = A_0,$$
$$\lim_{x=\infty} x[F(x) - A_0] = 0 = A_1,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\lim_{x=\infty} x^m\left[F(x) - \left(A_0 + \frac{A_1}{x} + \cdots + \frac{A_m}{x^m}\right)\right] = 0,$$

for all positive integral values of m. The given function $F(x)$ is then represented asymptotically by an expansion which is identically zero.

Now the derivative of this asymptotic expansion is likewise zero. Let us then consider the derived function. We have

$$D_x F(x) = F'(x) = e^{-x}\cos(e^{x^2})\cdot e^{x^2}\cdot 2x - e^{-x}\sin(e^{x^2})$$
$$= 2x\,e^{x^2-x}\cos(e^{x^2}) - e^{-x}\sin(e^{x^2}).$$

From this expression we have

$$\lim_{x=\infty} x^m F'(x) = \lim_{x=\infty} x^m\left\{2x\,e^{x^2-x}\cos(e^{x^2}) - e^{-x}\sin(e^{x^2})\right\}$$
$$\neq 0,$$

for <u>all</u> values of m. The value of $F'(x)$ increases indefinitely with x, as does also the value of $x^m F'(x)$. Hence we have here an asymptotically expansible function whose derivative possesses no such expansion. We have thus shown that <u>it is not always permissible to differentiate an asymptotic series</u>

It is apparent that there do exist functions the first derivatives of which do possess asymptotic expansions which are the term-wise derivatives of the asymptotic expansions of the given functions. We need only consider the derivative of the integral of an asymptotically expansible function. By the theorem of the previous article, if $f(x)$ possesses an asymptotic expansion, then does also the integral of $f(x)$, say $F(x)$; consequently, the derivative of $F(x)$, which is $f(x)$, possesses an asymptotic expansion. Moreover, since the asymptotic expansion of $F(x)$ is the term-wise integral of the asymptotic expansion of $f(x)$, and since this latter expansion is the term-wise derivative of the term-wise integral of itself, it follows that, if the derivative of an asymptotically expansible function possesses an asymptotic expansion, that asymptotic expansion is the term-wise derivative of the asymptotic expansion of the given function.

We shall now consider a function which is such that both it and its first derivative possess asymptotic expansions. Whittaker in "Modern Analysis" p. 169, gives the series

$$\frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \cdots$$

as the asymptotic expansion of the function

$$F(x) = \int_0^x \frac{e^{t-x} - e^{-x}}{t}\, dt,$$

when x is real and positive.

According to the rule for differentiating under the integral sign, we have

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha} + \int_a^b \left\{\frac{\partial}{\partial\alpha} f(x,\alpha)\right\} dx,$$

which becomes in our case, letting

$$f(t;x) = \frac{e^{t-x} - e^{-x}}{t} = \frac{e^{-x}(e^t - 1)}{t},$$

$$F'(x) = \frac{d}{dx}\int_0^x f(t,x)\,dt = f(x,x)\frac{dx}{dx} - f(0,x)\frac{d0}{dx} + \int_0^x \left\{\frac{\partial}{\partial x} f(t,x)\right\} dt$$

$$= \frac{e^{x-x} - e^{-x}}{x} + \int_0^x \left\{\frac{\partial}{\partial x}\left(\frac{e^{t-x} - e^{-x}}{t}\right)\right\} dt$$

$$= \frac{1 - e^{-x}}{x} + \int_0^x \frac{e^{-x}(1 - e^t)}{t}\, dt$$

$$= \frac{1 - e^{-x}}{x} - \int_0^x \frac{e^{t-x} - e^{-x}}{t}\, dt$$

$$= \frac{1 - e^{-x}}{x} - F(x).$$

Now let us examine the asymptotic expansion of this new function. Since $F'(x)$ is the difference between $\frac{1 - e^{-x}}{x}$ and $F(x)$, and since the last function possesses an asymptotic expansion, we need only examine the asymptotic expansion of $\frac{1 - e^{-x}}{x}$ and apply the theorem already known to the result.

Let $$\varphi(x) = \frac{1 - e^{-x}}{x} = \frac{1 - \frac{1}{e^x}}{x} = \frac{1}{x} - \frac{1}{e^x}$$

Then $\lim_{x=\infty} \phi(x) = \lim_{x=\infty} \left[\frac{1}{x} - \frac{1}{e^x}\right] = 0 = A_0$;

from this we can see that the asymptotic expansion of $\phi(x)$ is identically zero. Therefore, by virtue of the Subtraction Theorem, the asymptotic expansion of $F'(x)$ exists and is the difference between the asymptotic expansions of the functions $\phi(x)$ and $f(x)$.

Now the derivative of the asymptotic expansion of $F(x)$ gives

$$\frac{d}{dx}\left[\frac{1}{x} + \frac{1!}{x^2} + \frac{2!}{x^3} + \cdots + \frac{(n-1)!}{x^n} + \cdots\right] = \sum \frac{-n(n-1)!}{x^{n+1}}$$

$$= -\sum \frac{n!}{x^{n+1}}.$$

But this last expression is exactly of the form in which the asymptotic expansion of $F(x)$ may be written. The truth of the statement made above is thus illustrated by an example.

Insofar as the general statement (See p. 68) is concerned it appears that we must test each individual case in order to determine if the derivative of an asymptotically expansible function possesses an asymptotic expansion. That this is a rather unsatisfactory condition in which to leave the subject is at once apparent. The problem: To determine a set of (i) sufficient, (ii) necessary, (iii) necessary and sufficient, conditions that the derivative of a given asymptotically expansible function possess an asymptotic expansion.

is one to which I have given considerable attention but which I have been unable to solve satisfactorily. If $f(x)$ is a typical or general function of a class of asymptotically expansible functions, and if the derivative of $f(x)$ belongs to the same class of functions or is some such combination of functions of this class and of other asymptotically expansible functions as would bring it under one of the above established theorems on operations with asymptotic series, then we are generally able to determine by inspection the possibility of the asymptotic expansion of the derived function. Such will be found to be the case with the Bessel functions and some of their derivatives, to be treated in a later section.

Chapter III

Examples Of Asymptotically Expansible Functions

38. In the present chapter we shall consider some additional examples of asymptotically expansible functions. In some instances the more or less detailed development of the expansion will be given whereas in other cases only the final result will be stated. The Bessel functions are of such considerable importance in applied mathematics as to justify, I hope, their somewhat extensive treatment given below.

39. Formulas for Bessel's Functions. — Bessel's differential equation of order n

$$\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx} + \left(1 - \frac{n^2}{x^2}\right) z = 0$$

has for a particular solution

$$z = J_n(x)$$

where, if n is unrestricted in value,

$$(1)\quad J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2 \cdot 1!\,(n+1)} + \frac{x^4}{2^4 \cdot 2!\,(n+1)(n+2)} - \frac{x^6}{2^6 \cdot 3!\,(n+1)(n+2)(n+3)} + \frac{x^8}{2^8 \cdot 4!\,(n+1)(n+2)(n+3)(n+4)} - \cdots\right],$$

and $J_n(x)$ is called a Cylindrical Harmonic or Bessel's Function of the n^{th} order.*

* See Byerly, Fourier's Series and Spherical Harmonics, (1902) p. 219.

If n is any positive integer, then

$$(2) \qquad J_n(x) = \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^{n+2r} \frac{(-1)^r}{r!\,(n+r)!};$$

and if n is any negative integer, equal to $-s$, then

$$(3) \qquad J_n(x) = \sum_{r=s}^{\infty} \left(\frac{x}{2}\right)^{2r-s} \frac{(-1)^r}{r!\,(r-s)!} = \sum_{t=0}^{\infty} \left(\frac{x}{2}\right)^{s+2t} \frac{(-1)^{s+t}}{(s+t)!\;t!};$$

consequently, when n is any integer,

$$(4) \qquad J_n(x) = (-1)^n J_{-n}(x).$$

Whether n be a positive or a negative integer,* we may write

$$(5) \qquad J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}.\ \dagger$$

For all values of x, and for all values of n whose real part is greater than $-\frac{1}{2}$, the formula‡

$$J_n(x) = \frac{x^n}{2^n \cdot \Gamma(\frac{1}{2}) \cdot \Gamma(n+\frac{1}{2})} \int_0^{\pi} \cos(x\cos\phi)\, \sin^{2n}\phi\, d\phi,$$

or (6) $$J_n(x) = \frac{x^n}{2^n \sqrt{\pi}\, \Gamma(n+\frac{1}{2})} \int_0^{\pi} \sin^{2n}\phi \cos(x\cos\phi)\, d\phi,$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, is true.

If n is a positive integer (6) reduces to

$$(7) \qquad J_n(x) = \frac{1}{\pi} \frac{x^n}{1\cdot 3\cdot 5 \cdots (2n-1)} \int_0^{\pi} \sin^{2n}\phi \cos(x\cos\phi)\, d\phi;$$

and if $n = 0$, we have from (6) or (7)

$$(8) \qquad J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x\cos\phi)\, d\phi.$$

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*. See Whittaker, Modern Analysis, (1902), pp. 267, 274.

†. This formula is true for all values of n and x.

‡. Whittaker, loc. cit., p. 282. Referred to hereafter as Whittaker.

The following set of formulas, (9) – (13), are true for all values of n, real or complex*

(9) $$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x),$$

(10) $$\frac{d J_n(x)}{d x} = \frac{n}{x} J_n(x) - J_{n+1}(x),$$

(11) $$\frac{d J_n(x)}{d x} = \frac{1}{2}\left\{ J_{n-1}(x) - J_{n+1}(x)\right\},$$

(12) $$\frac{d J_n(x)}{d x} = J_{n-1}(x) - \frac{n}{x} J_n(x),$$

(13) $$\frac{J_{n+r}(x)}{x^{n+r}} = (-1)^r \frac{d^r}{(x\,dx)^r}\left\{\frac{J_n(x)}{x^n}\right\}, \quad (r \text{ a positive integer}).$$

When the order of a Bessel function $J_n(x)$ is a number of the form $k+\frac{1}{2}$, k any positive integer, then the Bessel function

$$J_{k+\frac{1}{2}}(x) = \frac{x^{k+\frac{1}{2}}}{2^{k+\frac{1}{2}}\Gamma(k+\frac{3}{2})}\left\{1 - \frac{x^2}{2(2k+3)} + \frac{x^4}{2\cdot 4\cdot(2k+3)(2k+5)} - \cdots\right\}$$

may be expressed in the following form:†

$$J_{k+\frac{1}{2}}(x) = (-2)^k (x)^{k+\frac{1}{2}} \frac{d^k}{d(x^2)^k}\left\{\frac{J_{\frac{1}{2}}(x)}{x^{\frac{1}{2}}}\right\}$$

and since

$$J_{\frac{1}{2}}(x) = \frac{2^{\frac{1}{2}} x^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\left\{1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 5} - \cdots\right\} = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x,$$

we have

(14) $$J_{k+\frac{1}{2}}(x) = \frac{(-1)^k (2x)^{k+\frac{1}{2}}}{\pi^{\frac{1}{2}}} \frac{d^k}{d(x^2)^k}\left(\frac{\sin x}{x}\right).$$

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* Whittaker, p. 275.

† Whittaker, p. 288

40. The Asymptotic Expansion of Bessel Functions. - The power series which have been obtained for $J_n(x)$ are convergent for all finite values of x, but they converge so slowly for large values of x, that they are practically useless for numerical calculation when the modulus of x is even moderately large. On account of their importance in physical investigations, considerable attention has been given to securing for these functions more rapidly-converging series which would give good approximations with little labor. These investigations led to the asymptotic expansions of $J_n(x)$. Poisson* discussed the approximate value of $J_0(x)$ when x is large but not in a very detailed or satisfactory manner; and Jacobi† investigated the question for general integral values of n. The methods of Stokes‡ are important as being applicable to a large number of functions of a kind which frequently occur in physical investigations.

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* Poisson, Journal de l'École Polytechnique, tome 19, 1823, pp. 349 et seq.

† Jacobi, Gesammelte Werke, vol. 7, p. 174 et seq.

‡ Stokes, Camb. Phil. Trans., vol. 9, 1856, pp. 166 et seq.; also Mathematical and Physical Papers, vol. IV, 1904, pp. 77 et seq.; also Gray and Mathews, Treatise on Bessel Functions, 1895, pp. 34 et seq.

Others who have investigated these expansions include Hankel*, Hurwitz†, Weber‡, Lipschitz†, Jordan** and Nielsen§. Further references will be found in the "Encyklopädie"§§ and in Nielsen's text book.§ We shall here consider only the asymptotic expansion of $J_n(x)$ for positive real values of x, and shall follow very closely the treatment given by Whittaker.***

We shall make use of the integral-expression

$$J_n(x) = \frac{1}{(2\pi x)^{\frac{1}{2}}\Gamma(n+\frac{1}{2})} \cdot \left[\cos\left\{x-(n+\tfrac{1}{2})\tfrac{\pi}{2}\right\}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left\{\left(1+\frac{iu}{2x}\right)^{n-\frac{1}{2}}+\left(1-\frac{iu}{2x}\right)^{n-\frac{1}{2}}\right\}du\right.$$
$$\left.+\sin\left\{x-(n+\tfrac{1}{2})\tfrac{\pi}{2}\right\}\int_0^\infty e^{-u}u^{n-\frac{1}{2}}\left\{\left(1+\frac{iu}{2x}\right)^{n-\frac{1}{2}}-\left(1-\frac{iu}{2x}\right)^{n-\frac{1}{2}}\right\}i\,du\right],$$

which has been shown to furnish an approximate value of $J_n(x)$ for large positive values of x.§*

Let us now derive the asymptotic expansion of

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*Hankel, Mathematische Annalen, Vol. 1, pp. 457-501.
†Hurwitz, Mathematische Annalen, Vol. 33, p. 246.
‡Weber, Mathematische Annalen, Vol. 6, p. 148.
†Lipschitz, Crelle, vol. 56, pp. 189 et seq.
**Jordan, Cours d'Analyse, 1896, vol. 3, pp. 254-274.
§Nielsen, Handbuch der Cylinderfunctionen, 1904, pp. 156 &c.
§§Wangerin, Encyklopädie der Mathematischen Wissenschaften, Band 2, A, p. 748.
***Whittaker, Modern Analysis, 1902, pp. 292-294.
§*Whittaker, Modern Analysis, 1902, pp. 289, 290.

the typical integral involved in the above expression,

$$I = \int_0^\infty e^{-u} u^k \left(1 + \frac{iu}{2x}\right)^k du, \qquad (k > 0).$$

We have

$$\left(1 + \frac{iu}{2x}\right)^k = 1 + k\,\frac{iu}{2x} + \frac{k(k-1)}{2!}\left(\frac{iu}{2x}\right)^2 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}\left(\frac{iu}{2x}\right)^n + \frac{k(k-1)\cdots(k-n)}{n!}\int_0^{\frac{iu}{2x}} \left(\frac{iu}{2x} - t\right)^n (1+t)^{k-n-1} dt,$$

the continuation of the expansion being furnished by the integral. We therefore have

$$I = \int_0^\infty e^{-u} u^k du + \sum_{r=1}^{n} \frac{k(k-1)\cdots(k-r+1)}{r!}\left(\frac{i}{2x}\right)^r \int_0^\infty e^{-u} u^{k+r} du + \frac{k(k-1)\cdots(k-n)}{n!}\int_0^\infty e^{-u} u^k du \int_0^{\frac{iu}{2x}} \left(\frac{iu}{2x} - t\right)^n (1+t)^{k-n-1} dt,$$

whence

$$I = \Gamma(k+1)\left\{1 + \sum_{r=1}^{n} \frac{k(k-1)\cdots(k-r+1)}{r!}\left(\frac{i}{2x}\right)^r (k+r)(k+r-1)\cdots(k+1)\right\} + R_n,$$

where

$$(1) \qquad R_n = \frac{k(k-1)\cdots(k-n)}{n!}\left(\frac{i}{2x}\right)^{n+1} \int_0^\infty e^{-u} u^k du \int_0^u (u-v)^n \left(1 + \frac{iv}{2x}\right)^{k-n-1} dv.$$

If we assign to n any definite finite integral value, then as x is increased indefinitely, the remainder R_n tends to the limit

$$(2) \qquad \lim_{x=\infty} R_n = \frac{k(k-1)\cdots(k-n)}{n!}\left(\frac{i}{2x}\right)^{n+1} \int_0^\infty e^{-u} u^k du \int_0^u (u-v)^n dv,$$

since $\lim_{x=\infty} \left(1 + \frac{iv}{2x}\right)^{k-n-1} = 1;$

Now $\int_0^u (u-v)^n dv = \frac{u^{n+1}}{n+1},$

substituting which in (2) and combining, we have

$$(3)\qquad \lim_{x=\infty} R_n = \frac{k(k-1)\cdots(k-n)}{(n+1)!}\left(\frac{i}{2x}\right)^{n+1}\int_0^\infty e^{-u}u^{k+n+1}\,du$$

or finally, since $\int_0^\infty e^{-u}u^{k+n+1}\,du = \Gamma(k+n+2)$,

$$(4)\qquad \lim_{x=\infty} R_n = \left(\frac{i}{2}\right)^{n+1}\frac{k(k-1)\cdots(k-n)\,\Gamma(k+n+2)}{(n+1)!}\,\frac{1}{x^{n+1}}.$$

From (4) it follows that

$$\lim_{x=\infty} x^n R_n = 0,$$

and that therefore the function

$$I = \int_0^\infty e^{-u}u^k\left(1+\frac{iu}{2x}\right)^k du \qquad (k>0),$$

is represented asymptotically by the series

$$\Gamma(k+1)\left\{1+\sum_{r=1}^{\infty}\frac{(k+r)(k+r-1)\cdots(k-r+1)}{r!}\left(\frac{i}{2x}\right)^r\right\}.$$

Substituting this result in the expression for $J_n(x)$ with which we started, we have on simplifying

$$J_n(x)\sim\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left[\cos\left\{x-(n+\tfrac{1}{2})\tfrac{\pi}{2}\right\}\left\{1+\sum_{r=1}^{\infty}\frac{(n-\frac{1}{2}+2r)(n-\frac{3}{2}+2r)\cdots(n-2r+\frac{1}{2})}{(2r)!}\frac{(-1)^r}{(2x)^{2r}}\right\}\right.$$
$$\left.+\sin\left\{x-(n+\tfrac{1}{2})\tfrac{\pi}{2}\right\}\sum_{r=1}^{\infty}\frac{(n-\frac{3}{2}+2r)(n-\frac{5}{2}+2r)\cdots(n-2r+\frac{3}{2})}{(2r-1)!}\frac{(-1)^r}{(2x)^{2r-1}}\right],$$

this asymptotic expansion being valid for large positive real values of x.

It is at once seen that for very large positive real values of x, both Σ's become small and approach the limit zero when x becomes infinitely large; thus

for very large positive real values of x, the function $J_n(x)$ approximates to the value

$$\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left\{x - (n+\tfrac{1}{2})\frac{\pi}{2}\right\}.$$

If <u>n is an even integer</u>, this result becomes

$$(-1)^{\frac{n}{2}}\left(\frac{1}{\pi x}\right)^{\frac{1}{2}}(\cos x + \sin x),$$

while, if <u>n is an odd integer</u>, it becomes

$$(-1)^{\frac{n+1}{2}}\left(\frac{1}{\pi x}\right)^{\frac{1}{2}}(\cos x - \sin x).$$

Even when x is not very large, the value of $J_n(x)$ can be computed with very great accuracy and comparatively little labor from the asymptotic expansion. For all values of x greater than 10, the first three terms of the asymptotic expansion give the value of $J_0(x)$ and $J_1(x)$ correct to six places of decimals. The convergent series expansion for $J_0(10)$ is

$$J_0(10) = \frac{1}{1}\left[1 - \frac{10^2}{2^2\cdot} + \frac{10^4}{2^4\cdot 2!\ 2!} - \frac{10^6}{2^6\ 3!\ 3!} + \frac{10^8}{2^8\cdot 4!\ 4!} - \frac{10^{10}}{2^{10} 5!\ 5!} + \cdots\right]$$

In this expansion the 6" term is equal in value to the 5" term, each being equal to approximately 679. The value of the 11" term is approximately 7.261, while that of the 16" term is approximately 0.00055. It is thus seen that in order to obtain the value of $J_0(10)$ from the ordinary convergent expansion it will be necessary to use the first 20 terms,

the value of the 14" term being approximately 0.0000004. It is, of course, evident that to secure a given approximation to the value of $J_0(x)$ as x increases, requires a greater and greater number of terms of the convergent expansion whereas not more than 3 terms of the ultimately divergent series which represents $J_0(x)$ asymptotically are required to secure a result to 6 decimals.

41. Asymptotic Expansions depending upon that of $J_n(x)$. — By making use of the asymptotic expansion of $J_n(x)$ and known formulas involving $J_n(x)$, we are enabled, by virtue of the theorems developed for operating with asymptotic series, to obtain some rather interesting and important results.

Let us designate by $S_{(k)}$ the asymptotic expansion of $J_k(x)$. Then since by (9), §39,

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x),$$

we have

$$J_{n-1}(x) \sim S_{(n-1)}, \quad J_{n+1}(x) \sim S_{(n+1)}, \quad J_n(x) \sim S_{(n)},$$

whence by virtue of the Addition Theorem, §20,

$$(1) \qquad \frac{2n}{x} J_n(x) \sim S_{(n-1)} + S_{(n+1)},$$

that is, we are enabled to obtain the asymptotic expansion of $J_n(x)$ from those of $J_{n-1}(x)$ and $J_{n+1}(x)$ by a very simple operation.

The asymptotic equality (1) furnishes an ex-

ample of the asymptotic expansion of the product of an algebraic function by an asymptotically expansible uniform integral function. It is not, however, generally true that the product of an algebraic function $f(x)$ by an asymptotically expansible function $J(x)$ possesses an asymptotic expansion.

Again, since by (11), § 39,

$$\frac{d J_n(x)}{dx} = \frac{1}{2}\left\{ J_{n-1}(x) - J_{n+1}(x)\right\},$$

we have, by virtue of the Subtraction Theorem, §2.1,

$$\frac{d J_n(x)}{dx} \sim \frac{1}{2}\left\{ S_{(n-1)} - S_{(n+1)}\right\}. \tag{2}$$

As another example of the asymptotic expansion of the derivative of an asymptotically expansible function, we have the formula*

$$16\,\frac{d^4 J_n(x)}{dx^4} = J_{n-4}(x) - 4\,J_{n-2}(x) + 6\,J_n(x) - 4\,J_{n+2}(x) + J_{n+4}(x),$$

whence we have the asymptotic equality

$$16\,\frac{d^4 J_n(x)}{dx^4} \sim S_{(n-4)} - 4\,S_{(n-2)} + 6\,S_{(n)} - 4\,S_{(n+2)} + S_{(n+4)}.$$

If in these two formulas we let $n = 0, 1, 2, 3, \cdots$, successively we have, remembering that

$$J_{-n}(x) = (-1)^n J_n(x), \qquad (n \text{ an integer})$$

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* Whittaker, Modern Analysis, 1902, p. 275

$$16\frac{d^4J_0(x)}{dx^4} = 6J_0(x) - 8J_2(x) + 2J_4(x);$$

$$16\frac{d^4J_1(x)}{dx^4} = 10J_1(x) - 5J_3(x) + J_5(x);$$

$$16\frac{d^4J_2(x)}{dx^4} = -4J_0(x) + 7J_2(x) - 4J_4(x) + J_6(x);$$

$$16\frac{d^4J_3(x)}{dx^4} = -5J_1(x) + 6J_3(x) - 4J_5(x) + J_7(x);$$

. . . . . ;

and

$$J_0^{(IV)}(x) \sim \tfrac{3}{8}S_{(0)} - \tfrac{1}{2}S_{(2)} + \tfrac{1}{8}S_{(4)};$$
$$J_1^{(IV)}(x) \sim \tfrac{5}{8}S_{(1)} - \tfrac{5}{16}S_{(3)} + \tfrac{1}{16}S_{(5)};$$
$$J_2^{(IV)}(x) \sim -\tfrac{1}{4}S_{(0)} + \tfrac{7}{16}S_{(2)} - \tfrac{1}{4}S_{(4)} + S_{(6)};$$
$$J_3^{(IV)}(x) \sim -\tfrac{5}{16}S_{(1)} + \tfrac{3}{8}S_{(3)} - \tfrac{1}{4}S_{(5)} + S_{(7)};$$

. . . . . .

Designating by $J_n^{(I)}, J_n^{(II)}, J_n^{(III)}, \ldots, J_n^{(t)}$ the 1st, 2nd, 3d, 4th, ..., t^{th} derivatives of $J_n(x)$, respectively, we obtain by a continued application of (11), §39,

$$J_n^{(I)} = \tfrac{1}{2}\left(J_{n-1} - J_{n+1}\right);$$

$$J_n^{(II)} = \tfrac{1}{2}\left(J_{n-1}^{(I)} - J_{n+1}^{(I)}\right) = \tfrac{1}{4}\left(J_{n-2} - 2J_n + J_{n+2}\right);$$

$$J_n^{(III)} = \tfrac{1}{2}\left(J_{n-1}^{(II)} - J_{n+1}^{(II)}\right) = \tfrac{1}{4}\left(J_{n-2}^{(I)} - 2J_n^{(I)} + J_{n+2}^{(I)}\right)$$
$$= \tfrac{1}{8}\left(J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}\right);$$

$$J_n^{(IV)} = \tfrac{1}{2}\left(J_{n-1}^{(III)} - J_{n+1}^{(III)}\right) = \tfrac{1}{4}\left(J_{n-2}^{(II)} - 2J_n^{(II)} + J_{n+2}^{(II)}\right)$$
$$= \tfrac{1}{8}\left(J_{n-3}^{(I)} - 3J_{n-1}^{(I)} + 3J_{n+1}^{(I)} - J_{n+3}^{(I)}\right)$$
$$= \tfrac{1}{16}\left(J_{n-4} - 4J_{n-2} + 6J_n - 4J_{n+2} + J_{n+4}\right);$$

. . . . . .

$$J_n^{(t)} = \tfrac{1}{2}\left(J_{n-1}^{(t-1)} - J_{n+1}^{(t-1)}\right) = \tfrac{1}{4}\left(J_{n-2}^{(t-2)} - 2J_n^{(t-2)} + J_{n+2}^{(t-2)}\right)$$
$$= \tfrac{1}{8}\left(J_{n-3}^{(t-3)} - 3J_{n-1}^{(t-3)} + 3J_{n+1}^{(t-3)} - J_{n+3}^{(t-3)}\right)$$

$$= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= \frac{1}{2^l}\left(J_{n-l} - l\,J_{n-l+2} + \frac{l(l-1)}{2}J_{n-l+4} - \cdots + (-1)^l J_{n+l}\right)$$

$$= \frac{1}{2^l}\sum_{r=0}^{l}(-1)^r\, {}_lC_r\; J_{n-l+2r}.$$

From the last equation we obtain the general asymptotic equality involving the l^{th} derivative of the Bessel function $J_n(x)$,

$$J_n^{(l)} \sim \frac{1}{2^l}\sum_{r=0}^{l}(-1)^r\, {}_lC_r\; S_{(n-l+2r)} \qquad (l=1,2,3,\cdots).$$

These results are of considerable importance since they enable us in conjunction with other properties of Bessel functions to determine, by the application of the established asymptotic expansion theorems, many classes of asymptotically expansible functions the derivatives of which possess asymptotic expansions which latter may be obtained either by the term-wise differentiation of the known asymptotic expansion or by performing simple operations upon the readily obtainable asymptotic expansions of closely related functions.

As another example, from the relation*

$$J_0^{(II)}(x) = J_2(x) + J_0^{(I)}(x),$$

since $\quad J_2(x) \sim S_{(2)}$, and $J_0^{(I)}(x) = -J_1(x) \sim -S_{(1)}$,

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* Whittaker, Modern Analysis, 1902, p. 275.

we have

$$J_0^{(II)} \sim S_{(2)} - S_{(1)}.$$

Again, from the identity *

$$2J_0^{(III)} = J_3 - 3J_0^{(I)} + \frac{3}{x} J_0^{(II)},$$

we have, making use of (9), §39,

$$\begin{aligned} 2J_0^{(III)} &= J_3 - 3J_0^{(I)} + \frac{3}{x} J_2 + \frac{3}{x} J_0^{(I)} \\ &= J_3 + 3J_1 + \frac{3}{x} J_2 - \frac{3}{x} J_1 \\ &= J_3 + 3J_1 + \frac{3}{4}\frac{4}{x} J_2 - \frac{3}{2}\frac{1}{x} J_1 \\ &= J_3 + 3J_1 + \frac{3}{4} J_1 + \frac{3}{4} J_3 - \frac{3}{2} J_0 - \frac{3}{2} J_2 \\ &= \frac{7}{4} J_3 - \frac{3}{2} J_2 + \frac{15}{4} J_1 - \frac{3}{2} J_0, \end{aligned}$$

whence we have the asymptotic equality

$$J_0^{(III)} \sim \frac{7}{8} S_{(3)} - \frac{3}{4} S_{(2)} + \frac{15}{8} S_{(1)} - \frac{3}{4} S_{(0)}.$$

The equation †

$$\{J_0(z)\}^2 + 2\{J_1(z)\}^2 + 2\{J_2(z)\}^2 + 2\{J_3(z)\}^2 + \cdots = 1,$$

gives when z is real

$$J_0(z) \not> 1, \quad \text{and} \quad J_n(z) \not> n^{-\frac{1}{2}} \quad (n > 0),$$

whence it follows that the sum of all the terms of the asymptotic expansion for $J_n(z)$ up to and including the term of least absolute value must be $\not> 1$ or $\not> \frac{1}{4}$ for $n = 0$ or $n > 0$.

For all values of μ and ν the product ‡ of $J_\mu(z)$ by $J_\nu(z)$ is

$$J_\mu(z) \cdot J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{\mu+\nu+2n}{n} \left(\frac{1}{2}z\right)^{\mu+\nu+2n}}{\Gamma(\mu+n+1)\,\Gamma(\nu+n+1)},$$

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* Whittaker, Modern Analysis, 1902, p. 275.
† Whittaker, p. 305. ‡ Whittaker, p 304. Ex. 2.

whence the asymptotic equality

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{\mu+\nu+2n}{n} \left(\frac{1}{2}z\right)^{\mu+\nu+2n}}{\Gamma(\mu+n+1)\,\Gamma(\nu+n+1)} \sim S_{(\mu)} \cdot S_{(\nu)}.$$

The equation*

$$J_{-\mu}(z)\, J_{\mu-1}(z) + J_{-\mu+1}(z) \cdot J_{\mu}(z) = \frac{2 \sin \mu\pi}{\pi z}$$

permits us to write the asymptotic expansion of $\frac{2 \sin \mu\pi}{\pi z}$,

$$\frac{2 \sin \mu\pi}{\pi z} \sim S_{(-\mu)} S_{(\mu-1)} + S_{(-\mu+1)} S_{(\mu)}.$$

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* See Whittaker, Modern Analysis, 1902, p. 305. Ex. 5. Due to Lommel.

Chapter IV.

Asymptotic Values Of Function.

4n. In this chapter we shall consider portions of an article by Le Roy* on "Valeurs Asymptotiques de certaines séries procédant suivant les puissances entières et positives d'une variable réele" as an illustration of his method of attack in developing a rule for determining the asymptotic value of certain given functions.

Let us suppose that the integral series (série entière)

$$(I) \qquad f(z) = \sum_{n=0}^{\infty} \alpha_n z^n,$$

wherein z is real, and all the α_n are positive, has a finite interval of convergence, which we shall suppose to be $(-1, +1)$, losing thereby nothing of generality. If $\sum_{n=0}^{\infty} \alpha_n$ then the numerical series is divergent, series (I) becomes infinite as $z = 1$. In this case it is proposed to study the asymptotic value of $f(z)$.

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*Le Roy, M. Édouard, Bulletin des Sciences Math. (1900) Ser. 2. V. 24. {245 / 268.

43. Case where $\alpha_n \geq 0$, but α_n constantly decreases and $\alpha_{n=\infty} \doteq 0+$. - The case where

$$\alpha_n \doteq A \qquad (A \neq 0),$$

may be at once discarded, since then

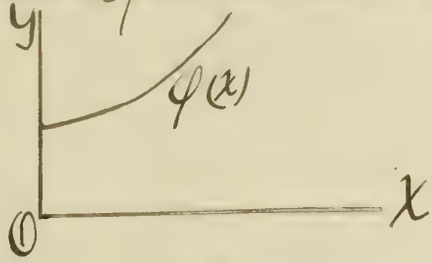
$$f(z) = \frac{A}{1-z} + \sum_{n=0}^{\infty} \beta_n z^n \qquad (\beta_{n=\infty} \doteq 0).$$

It is evidently sufficient that the conditions indicated be fulfilled by the α_n's beginning at some certain value of n. For simplicity let us suppose the conditions satisfied from the first term α_0.

To satisfy the required conditions, let us place

$$\alpha_n = e^{-\varphi(n)},$$

where $\varphi(x)$ is a continuous function, greater than zero, and increasing, such that the assumed quality holds for and after some value of n. Let us



begin to count from this value of n, i.e., let us suppose, in effect, that the equality holds from $n=0$, nothing in generality being lost thereby.

Let $\lim_{n=\infty} \frac{\varphi(n)}{n} = 0.$

Then series (I) $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$

is convergent for $|z|<1$*. If now the integral

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*See Chrystal's Algebra, Vol 2, p. 108, Ex. 2.

$$\int_0^\infty e^{-\varphi(x)}\,dx$$

is divergent, so also will be the series Σa_n by virtue of a theorem of Cauchy.

Under these conditions, $z = 1$ becomes a singular point of $f(z)$, for which $f(z)$ becomes infinite.

We now wish to study the behavior of $f(z)$ as $z \doteq 1$ by <u>real</u> values smaller than 1.

Let us place

$$z = e^{-\alpha},$$

$$\psi(x) = e^{-\varphi(x)-\alpha x} = e^{-\varphi(x)}\cdot e^{-\alpha x} = e^{-\varphi(x)}\cdot z^x,$$

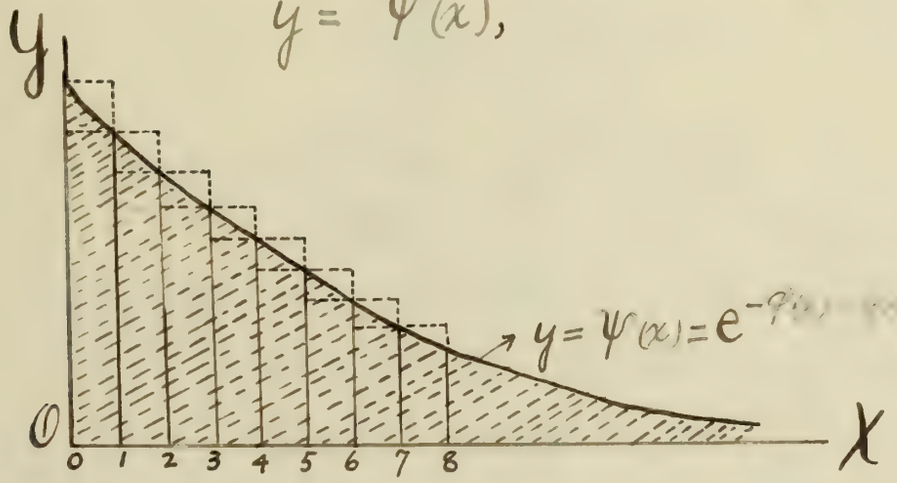
where α is <u>real</u> and <u>positive</u>.

Compare the value of the series

$$f(z) = \psi(0) + \psi(1) + \psi(2) + \cdots + \psi(n) + \cdots$$

with the area under the curve whose equation is

$$y = \psi(x),$$



which is given by the integral

$$J = \int_0^\infty e^{-\varphi(x)-\alpha x}\,dx$$
$$= \int_0^\infty e^{-\varphi(x)}\cdot z^x\,dx$$
$$= \int_0^\infty \psi(x)\,dx.$$

See Encyklopädie der mathematischen Wissenschaften, vol. 1, Part 1, p. 82

Using external rectangles of base 1 and altitude $\psi(n)$ we have clearly

$$J = \int_0^\infty \psi(x)\,dx < \psi(0) + \psi(1) + \psi(2) + \cdots + \psi(n) + \cdots,$$

i.e. $$J < f(z).$$

Using internal rectangles, we have

$$\psi(1) + \psi(2) + \psi(3) + \cdots + \psi(n) + \cdots < J,$$

i.e. $$f(z) - \psi(0) < J,$$

or $$f(z) < J + \psi(0).$$

Combining these two inequalities involving J and $f(z)$, we have $$J < f(z) < J + \psi(0);$$

whence $$1 < \frac{f(z)}{J} < 1 + \frac{\psi(0)}{J}.$$

Now $$\lim_{\alpha \doteq 0} \frac{\psi(0)}{J} = 0,$$

since $\psi(0)$ is constant, and $\lim_{\alpha \doteq 0} J = \infty$;

Then $$\lim_{\substack{\alpha \doteq 0 \\ \text{i.e. } z \doteq 1}} \frac{f(z)}{J} = 1,$$

whence the asymptotic equality

$$f(z) \sim J \qquad \text{as } \alpha \doteq 0, \text{ i.e. as } z \doteq 1.$$

We have thus reduced the calculation of $f(z)$ as $z \doteq 1$ to the calculation of the value of an asymptotically equivalent definite integral.

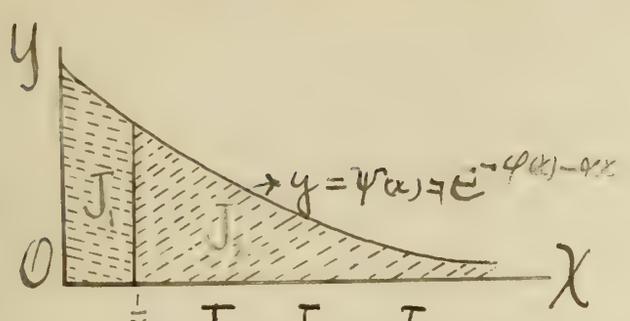
Let us now form the new integral

$$I = \int_0^\infty e^{-\Phi(x)}\,dx,$$

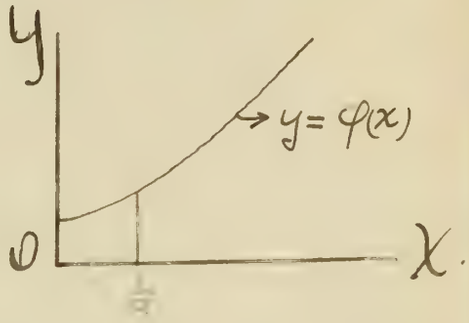
and form the ratio $\frac{J}{I}$.

We have now the following set of inequalities

(a) $$0 < \int_{\frac{1}{\alpha}}^{\infty} e^{-\varphi(x)-\alpha x}\,dx < e^{-\varphi(\frac{1}{\alpha})}\int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha x}\,dx < \frac{1}{\alpha}\,e^{-\varphi(\frac{1}{\alpha})}$$



$J = J_1 + J_2$.



For $$J_2 = \int_{\frac{1}{\alpha}}^{\infty} e^{-\varphi(x)-\alpha x}\,dx > 0.$$

Since $\varphi(x)$ is an increasing function, and since therefore
$$e^{-\varphi(x)} < e^{-\varphi(\frac{1}{\alpha})} \quad (\text{for } x > \tfrac{1}{\alpha} \text{ as in } J_2),$$
$$\therefore \int_{\frac{1}{\alpha}}^{\infty} e^{-\varphi(x)} e^{-\alpha x}\,dx < e^{-\varphi(\frac{1}{\alpha})}\int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha x}\,dx,$$

and $$e^{-\varphi(\frac{1}{\alpha})}\int_{\frac{1}{\alpha}}^{\infty} e^{-\alpha x}\,dx = \frac{1}{\alpha}e^{-1}e^{-\varphi(\frac{1}{\alpha})} = \frac{1}{e}\cdot\frac{1}{\alpha}\cdot e^{-\varphi(\frac{1}{\alpha})} < \frac{1}{\alpha}e^{-\varphi(\frac{1}{\alpha})};$$

whence we have inequality (a).

(b) $$\frac{1}{e}\int_0^{\frac{1}{\alpha}} e^{-\varphi(x)}\,dx < \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)-\alpha x}\,dx < \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)}\,dx.$$

For, $\because e^{-\alpha x}$ is a proper fraction, α and x both being > 0,
$$\therefore \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)-\alpha x}\,dx < \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)}\,dx.$$
Also $\because e^{-\alpha x} = \frac{1}{e^{\alpha x}} \doteq 1$, as $\alpha \doteq 0$, $\therefore e^{-\alpha x} > \frac{1}{e}$;
whence we have inequality (b).

(c) $$\int_0^{\frac{1}{\alpha}} e^{-\varphi(x)}\,dx > \frac{1}{\alpha}\,e^{-\varphi(\frac{1}{\alpha})}.$$

Since for $0 < x < \frac{1}{\alpha}$, $\quad 0 < \varphi(x) < \varphi(\frac{1}{\alpha})$,
$$\therefore e^{\varphi(x)} < e^{\varphi(\frac{1}{\alpha})} \text{ and } e^{-\varphi(x)} > e^{-\varphi(\frac{1}{\alpha})};$$
$$\therefore \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)}\,dx > \int_0^{\frac{1}{\alpha}} e^{-\varphi(\frac{1}{\alpha})}\,dx;$$
But $$\int_0^{\frac{1}{\alpha}} e^{-\varphi(\frac{1}{\alpha})}\,dx = e^{-\varphi(\frac{1}{\alpha})}\int_0^{\frac{1}{\alpha}} dx = \frac{1}{\alpha}\,e^{-\varphi(\frac{1}{\alpha})};$$
whence we have inequality (c).

Inequalities (a), (b) and (c) may be written

$$(a) \qquad 0 < J_2 < \frac{1}{\alpha} e^{-\varphi(\frac{1}{\alpha})};$$

$$(b) \qquad \frac{1}{e} I < J_1 < I;$$

$$(c) \qquad I > \frac{1}{\alpha} e^{-\varphi(\frac{1}{\alpha})}.$$

From these we see that

$$J_2 < I, \qquad J_1 < I, \text{ and } \because J = J_1 + J_2, \quad \therefore J < 2I.$$

$$J_2 > 0, \qquad J_1 > \frac{1}{e} I, \text{ and } \because J = J_1 + J_2, \quad \therefore J > \frac{1}{e} I.$$

From these latter inequalities we have

$$\frac{1}{e} I < J < 2I$$

which reduced gives

$$\frac{1}{e} < \frac{J}{I} < 2.$$

Thus, for α very small, J and I are of the same order of magnitude.

In the majority of cases it can be shown that

$$\lim_{\alpha = 0} \frac{J}{I} = A, \qquad (A \text{ being finite and} \neq 0).$$

To prove this we have the following:

$$J = \int_0^\infty e^{-\varphi(x) - \alpha x} dx.$$

$$\therefore \frac{\partial J}{\partial \alpha} = \int_0^\infty \left\{ \frac{\partial}{\partial \alpha} \left(e^{-\varphi(x) - \alpha x} \right) \right\} dx = \int_0^\infty e^{-\varphi(x) - \alpha x} \cdot -x \cdot dx.$$

$$I = \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)} dx.$$

$$\therefore \frac{\partial I}{\partial \alpha} = f(\tfrac{1}{\alpha}, \alpha) \frac{d\frac{1}{\alpha}}{d\alpha} - f(0, \alpha) \frac{d(0)}{d\alpha} + \int_0^{\frac{1}{\alpha}} \left\{ \frac{\partial}{\partial \alpha} e^{-\varphi(x)} \right\} dx$$

$$= e^{-\varphi(\frac{1}{\alpha})} \left(-\frac{1}{\alpha^2} \right) - 0 + 0$$

$$= -\frac{1}{\alpha^2} e^{-\varphi(\frac{1}{\alpha})}.$$

For "Differentiation under the integral sign," see Carr, Synopsis of Mathematics, p. 356, formulas 2253, 2255, 2257.

$$\therefore \quad \frac{\partial J}{\partial I} = \alpha^2 e^{\varphi(\frac{1}{\alpha})} \int_0^\infty e^{-\varphi(x)-\alpha x} \cdot x \cdot dx.$$

Now place $\alpha x = y$, $\frac{1}{\alpha} = \beta$.

Then $x = \beta y$, $dx = \beta\, dy$,

and when $\left\{ \begin{matrix} x = 0 \\ x = \infty \end{matrix} \right.$, $\left. \begin{matrix} y = 0 \\ y = \infty \end{matrix} \right\}$,

whence we have on substituting and simplifying,

$$\frac{\partial J}{\partial I} = \frac{1}{\beta^2} e^{\varphi(\beta)} \int_0^\infty e^{-\varphi(\beta y)-y} \cdot \beta y \cdot \beta \cdot dy$$

$$= \int_0^\infty e^{\varphi(\beta)-\varphi(\beta y)} \cdot e^{-y} \cdot y\, dy,$$

of which expression it is generally easy to obtain the limit as $\beta = \infty$, that is, as $\alpha \doteq 0$.

For example, if $x\varphi'(x)$ is a decreasing function of x, $\int_0^1 e^{\varphi(\beta)-\varphi(\beta y)} \cdot e^{-y} \cdot y\, dy$ decreases (as β increases), while remaining $> \int_0^1 e^{-y} \cdot y\, dy$, which equals $1 - \frac{2}{e}$;

also $\int_1^\infty e^{\varphi(\beta)-\varphi(\beta y)} e^{-y} \cdot y\, dy$ increases (as β increases), while remaining $< \int_1^\infty e^{-y} \cdot y\, dy$, which equals $\frac{2}{e} - \frac{1+\infty}{e^\infty} \doteq \frac{2}{e}$.

For example, let

$$\alpha_n = \frac{1}{n^p} \qquad (0 < p < 1).$$

Then $e^{\varphi(x)} = x^p$; $e^{\varphi(\beta)} = \beta^p$; $e^{-\varphi(\beta y)} = \beta^{-p} \cdot y^{-p}$;

whence $\lim_{\left\{ \begin{matrix} \alpha \doteq 0 \\ \beta = \infty \end{matrix} \right.} \frac{J}{I} = \int_0^\infty e^{\varphi(\beta)-\varphi(\beta y)} \cdot e^{-y} \cdot y \cdot dy = \int_0^\infty \beta^p \cdot \beta^{-p} y^{-p} \cdot e^{-y} y \cdot dy$

$$= \int_0^\infty y^{1-p} \cdot e^{-y}\, dy$$

$$= \Gamma(2-p).$$

Now, $J = \int_0^\infty e^{-\varphi(x)-\alpha x} dx = \int_{x=0}^{x=\infty} x^{-p} \cdot e^{-\alpha x} \cdot dx$

$$= \int_{y=0}^{y=\infty} \beta^{-p} y^{-p} e^{-y} \cdot \beta \cdot dy$$

$$= \beta^{1-p} \int_0^\infty e^{-y} \cdot y^{1-p} dy$$

$$= \beta^{1-p} \cdot \Gamma(1-p).$$

Now since $f(z) \sim J$, and since $z = e^{-\alpha}$, and therefore $z \doteq 1$ as $\alpha \doteq 0$, i.e. as $\beta = \infty$, and since therefore $\beta = \infty$ at the same time as $\frac{1}{1-z}$, then we have the asymptotic equality

$$f(z) \sim \frac{\Gamma(1-p)}{(1-z)^{1-p}}.$$

In order to calculate I, we may evidently choose whatever inferior limit we like, since any finite number of terms of an infinite series may be omitted without affecting the summability of that series. Moreover we may in I replace $e^{-\varphi(x)}$ by another expression whose ratio to $e^{-\varphi(x)}$ approaches the limit 1 as x becomes infinite. Let $\chi(x)$ be such an expression. Then

$$\lim_{\alpha \doteq 0} \frac{I = \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)} dx}{\int_0^{\frac{1}{\alpha}} \chi(x)\, dx} = \frac{\frac{\partial}{\partial \alpha} \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)} dx}{\frac{\partial}{\partial \alpha} \int_0^{\frac{1}{\alpha}} \chi(x)\, dx}$$

$$= \frac{-\frac{1}{\alpha^2} \cdot e^{-\varphi\left(\frac{1}{\alpha}\right)}}{-\frac{1}{\alpha^2} \chi\left(\frac{1}{\alpha}\right)}$$

$$= \frac{e^{-\varphi\left(\frac{1}{\alpha}\right)}}{\chi\left(\frac{1}{\alpha}\right)}.$$

$$\doteq 1 \text{ (by hypothesis).}$$

This enables us to considerably simplify the calculation of I in many cases. Thus, if we have

$$\alpha_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots 2n},$$

we may, $\because \varphi(x) = e^{-\varphi(x)}$, write

$$e^{-\varphi(x)} = \frac{1 \cdot 3 \cdot 5 \cdots (2x-1)}{2 \cdot 4 \cdot 6 \cdots 2x} \times \frac{2 \cdot 4 \cdot 6 \cdots 2x}{2 \cdot 4 \cdot 6 \cdots 2x}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4 \cdots 2x}{2^x(1\cdot 2\cdot 3\cdots x)\cdot 2^x(1\cdot 2\cdot 3\cdots x)}$$

$$= \frac{(2x)!}{2^{2x}\,(x!)^2}$$

$$= \frac{\Gamma(2x+1)}{2^{2x}\{\Gamma(x+1)\}^2};$$

and, $\therefore\ \chi(x) \doteq e^{-\varphi(x)}$, and $\lim_{x=\infty} x! = \sqrt{2\pi x}\cdot\left(\frac{x}{e}\right)^x$,* we may also write

$$\chi(x) = \frac{(2x)!}{2^{2x}\,(x!)^2} = \frac{\sqrt{4\pi x}\,(2x)^{2x}}{e^{2x}2^{2x}\cdot\sqrt{2\pi x}\cdot\sqrt{2\pi x}\cdot x^{2x}}$$

$$= \frac{1}{\sqrt{\pi x}}.$$

For a further simplification in the process of calculating the asymptotic value of $f(z)$, we have,† $\because z = e^{-\alpha}$,

$$\frac{1}{1-z} - \frac{1}{\alpha} = \frac{e^\alpha}{e^\alpha - 1} - \frac{1}{\alpha}$$

$$= \frac{1}{2} + \frac{B_1}{2!}\alpha - \frac{B_2}{4!}\alpha^3 + \frac{B_3}{6!}\alpha^5 \cdots + (-1)^p \frac{B_{p+1}}{(2p+2)!}\alpha^{2p+1} + \cdots$$

for $|\alpha| < 2\pi$, wherein $B_1, B_2, \ldots$ are the numbers of Bernoulli.§ Therefore

$$\int_0^{\frac{1}{1-z}} e^{-\varphi(x)}\,dx - \int_0^{\frac{1}{\alpha}} e^{-\varphi(x)}\,dx \doteq 0 \quad \text{as } \alpha \doteq 0.$$

We may consequently replace

$$I = \int_0^{\infty} e^{-\varphi(x)}\,dx \quad \text{by} \quad \int_0^{\frac{1}{1-z}} e^{-\varphi(x)}\,dx.$$

---

* Otherwise this is: The asymptotic value of $\Gamma(x+1)$ is $\sqrt{2\pi}\,e^{-x}\,x^{x+\frac{1}{2}}$.

† Chrystal's Algebra, Vol. II, p. 344.

‡ For this expansion, see E. Borel, Leçons sur les Séries Divergentes, p. 23.

§ $B_p = 2\dfrac{(2p)!}{(2\pi)^{2p}}\left(1 + \dfrac{1}{2^{2p}} + \dfrac{1}{3^{2p}} + \dfrac{1}{4^{2p}} + \cdots\right)$.

in the calculation of the asymptotic value of $f(z)$.

Collecting the results of this article we are able to state the following

Theorem. Let $\alpha(x)$ be a function such that

$$\alpha_n = \alpha(n).$$

Let $\beta(x)$ be a function whose derivative $\beta'(x)$ is such that the ratio

$$\frac{\beta'(x)}{\alpha(x)} \doteq 1 \quad \text{as} \quad x = \infty.$$

Then the asymptotic value of $f(z)$ for z in the neighborhood of 1 is of the form

$$C\,\beta\left(\frac{1}{1-z}\right) \qquad (C = \text{constant}),$$

provided the positive coefficients α_n continue to decrease and $\alpha_n \doteq 0$ as $n = \infty$.

We have

$$\alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_n = \int_0^n \alpha(x)\,dx + C + \epsilon_n,$$

C being some constant and $\epsilon_n \doteq 0$, as $n = \infty$.* Then the series $\Sigma \alpha_n$ becomes infinite in value with n as does the integral

$$\int_0^n \alpha(x)\,dx.$$

Consequently, $f(z)$ increases indefinitely, when $z \doteq 1$, in the same manner that the series of coefficients diverges.

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* If $\alpha_n = \frac{1}{n}$, $C = \sum_1^\infty \frac{1}{n} - \int_1^n \frac{1}{x}\,dx = \sum_1^n \frac{1}{n} - \log n = 0.57721\ldots$, and this number C is called Euler's constant. See Chrystal's Algebra, Vol. II, p. 81; also Cantor's Vorlesungen über Geschichte der Mathematik, Vol. III, pp. 662–665.

Briefly, there exists an intimate relation among the principal value of α_n for $n = \infty$, the manner of divergence of the series $\sum_0^\infty \alpha_n$, and the order of infinitude of $f(z)$ for $z = 1$.

44. Applications of Rule to Examples.

Example 1. Given $\alpha_n = \frac{1}{n^p}$, $(0 < p < 1)$; find the asymptotic value of $f(z)$ in the neighborhood of 1.

Solution. Here

$$\sum_1^\infty \alpha_n = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots,$$

which is a divergent series for $p \leqq 1$.

$$\alpha(x) = \frac{1}{x^p} = x^{-p}.$$

$$\beta(x) = \int \beta'(x)\,dx = \int x^{-p}\,dx = \frac{1}{1-p}\,x^{1-p}.$$

$$C\,\beta\left(\frac{1}{1-z}\right) = \frac{C}{1-p}\,\frac{1}{(1-z)^{1-p}} = K\,\frac{1}{(1-z)^{1-p}}.$$

$$\therefore \quad f(z) \sim K\,\frac{1}{(1-z)^{1-p}}.$$

Frequently in writing the asymptotic value of $f(z)$, a constant factor will be omitted.

Example 2. Given

$$\alpha_n = \frac{(Ln)^p}{n^q}, \quad (p > -1, \quad 0 < q < 1), \quad Ln = \log n;$$

find the asymptotic value of $f z = \Sigma \alpha_n z^n$ as $z = 1$.

Solution. Here

$$\alpha(x) = (Lx)^p \cdot x^{-q}.$$

$$\beta(x) = \int \alpha(x)\,dx = \int x^{-q}(Lx)^p\,dx$$

$$= \frac{x^{1-q}}{1-q}(Lx)^p - \frac{p}{1-q}\int x^{-q}(Lx)^{p-1}\,dx,$$

obtained on integrating by parts, letting

$$u = (Lx)^p \text{ and } dv = x^{-q}\,dx.$$

Continuing the integration by parts, we have

$$\beta(x) = \frac{x^{1-q}(\mathrm{L}\,x)^p}{1-q}\left[1 - \frac{p}{1-q}(\mathrm{L}\,x)^{-1} + \frac{p(p-1)}{(1-q)^2}(\mathrm{L}\,x)^{-2} - \cdots + (-1)^p \frac{p!}{(1-q)^p}(\mathrm{L}\,x)^{-p}\right]$$

$$\doteq \frac{x^{1-q}(\mathrm{L}\,x)^p}{1-q},$$

as $x \doteq \infty$, since the limit, as x becomes infinite, of the expression in the brackets, is 1.

Then, by the rule,

$$f(z) \sim C\,\beta\left(\frac{1}{1-z}\right) \sim \frac{C}{1-q}\cdot\frac{\left(\mathrm{L}\,\frac{1}{1-z}\right)^p}{(1-z)^{1-q}}.$$

This last result may be written

$$f(z) \sim \frac{\left(\mathrm{L}\,\frac{1}{1-z}\right)^p}{(1-z)^{1-q}}.$$

Example 3. Find the asymptotic value of $f(z) = \sum \alpha_n z^n$ as $z \doteq 1$, given that

$$\alpha_n = \left[\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\right]^p \qquad (0<p<2).$$

Solution. By the reduction made on p. 94, we have

$$\alpha(x) = \left[\frac{1\cdot 3\cdot 5\cdots(2x-1)}{2\cdot 4\cdot 6\cdots(2x)}\right]^p \sim \left[\frac{1}{\sqrt{\pi x}}\right]^p$$

Here then we replace $\alpha(x)$ by $\chi(x)$, an asymptotically equivalent expression, without affecting the asymptotic value of $f(z)$.* We have

$$\beta(x) = \int \pi^{-p/2} x^{-p/2}\,dx = \frac{1}{(1-p/2)\,\pi^{p/2}}\,x^{1-p/2} = C_1\,x^{1-p/2}.$$

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* See discussion on page 93.

Since $f(z) \sim C\ 3\left(\frac{1}{1-z}\right)$, we have

$$f(z) \sim K \frac{1}{(1-z)^{1-\frac{p}{2}}}$$

or simply $$f(z) \sim \frac{1}{(1-z)^{1-\frac{p}{2}}}$$

It is to be noted that in this example the series of coefficients $\sum_0^\infty \alpha_n$ is divergent. For the series is

$$\sum_1^\infty \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right]^p = \frac{1}{2} + \frac{3/2}{4} + \frac{15/8}{6} + \frac{105/48}{8} + \frac{945/384}{10} + \cdots$$

$$= \frac{1/2}{1} + \frac{3/4}{2} + \frac{15/16}{3} + \frac{105/96}{4} + \frac{945/768}{5} + \cdots,$$

which series is, term for term after the third term, greater than the known divergent harmonic series $\sum_1^\infty \frac{1}{n}$. A similar statement can be made regarding the series of coefficients of the previous and following example.

Example 4. Find the asymptotic value of $f(z)$, given

$$\alpha_n = \frac{1}{n\cdot L n\cdot L^{(2)}n\cdot L^{(3)}n\cdot\ldots\cdot L^{(p-1)}n\cdot\left[L^{(p)}n\right]^q},$$

where $L^{(p)}n = \operatorname{Log} L^{(p-1)}n$, $q \leqq 1$.

Solution. Here the series $\sum_1^\infty \alpha_n$ is divergent* when, and only when, $q \leqq 1$. In order to calculate $\beta(x) = \int \alpha(x)\,dx$, we have the following:

$$\frac{d}{dx}(\log\log\log\log x) = \frac{d}{dx}\left(L^{(4)}x\right) = \frac{1}{x\cdot L x\cdot L^{(2)}x\cdot L^{(3)}x},$$

and in general,

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*See Borel, Leçons sur les séries a termes positifs, 1902, p. 6.

$$\frac{d}{dx}\left(L^{(n)}x\right) = \frac{1}{x\cdot Lx\cdot L^{(2)}x\cdots L^{(n-1)}x};$$

also $$\frac{d}{dx}\left(L^{(n)}x\right)^m = m\left(L^{(n)}x\right)^{m-1}\frac{d}{dx}\left(L^{(n)}x\right)$$

$$= \frac{m}{x\cdot Lx\cdot L^{(2)}x\cdots L^{(n-1)}x\cdot\left[L^{(n)}x\right]^{m-1}}.$$

From this it follows at once that

$$\beta(x) = \int \frac{dx}{x\cdot Lx\cdot L^{(2)}x\cdots L^{(p-1)}x\cdot\left[L^{(p)}x\right]^q} = \frac{\left(L^{(p)}x\right)^{1-q}}{1-q}.$$

Now since we have $f(z) \sim C\cdot\beta\left(\frac{1}{1-z}\right)$,

$$f(z) \sim K\left[L^{(p)}\frac{1}{1-z}\right]^{1-q},$$

where $K = \frac{C}{1-q}$, and may be omitted without vitiating the result.

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